Homological and homotopical constructions for functors on ordered groupoids

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Abstract

The main topic of this thesis is the generalization to ordered groupoids of some results and constructions that have arisen in groupoid theory and its applications in homological and homotopical algebra. We study fibrations of ordered groupoids, and show that the covering homotopy property and star-surjectivity are not equivalent properties. We establish some formal properties of functors having these properties, and define a new quotient construction for ordered groupoids that leads to a factorization of any functor of ordered groupoids as a star-surjective followed by a star-injective functor. We give a direct proof of Ehresmann's Maximum Enlargement Theorem. Coupled with our quotient construction, The Maximum Enlargement Theorem gives a universal factorization of any functor of ordered groupoids as a fibration followed by an enlargement followed by a covering. We construct the mapping cocylinder M^{ϕ} of an ordered functor $\phi : G \to H$, and show directly that the morphism $M^{\phi} \to H$ has the covering homotopy property. We construct the derived module D_{θ} of an ordered functor and use it to study two adjoint functors between the category of ordered crossed complexes and the category of ordered chain complexes. Finally, we consider the groupoid of derivations of crossed modules of groups and of ordered groupoids, and in the latter case we use semiregular crossed modules to derive results on homotopies and endomorphisms.

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Chapter 1

Introduction

The idea of treating a groupoid – a small category in which every arrow is invertible – as an algebraic structure has been established as a profitable approach to problems in group theory, homotopy theory, and inverse semigroup theory. The book [18] by P J Higgins is an excellent introduction, and the homotopy theory applications are treated in [10]. The applications of groupoids in describing symmetry is concisely surveyed in [31]. C. Ehresmann developed a new foundation of differential geometry based on groupoids [12]. His algebraic model of a pseudogroup of transformations was an *ordered* groupoid. The introduction of an ordered structure leads to a separation from group theory since a group, when considered as an ordered groupoid, can only carry the equality ordering. Instead, the most closely related algebraic structure is that of an *inverse semigroup*. Inverse semigroups correspond to a certain subclass of ordered groupoids, called *inductive* groupoids, in which the poset of identity elements forms a semilattice. The interactions between the theories of ordered groupoids and inverse semigroups are a major theme of the book [21] of M V Lawson.

The algebra of ordered groupoids been studied independently, as in [20] and [30]. In these papers, the emphasis is on properties of functors between ordered groupoids. However, some of this development has by-passed earlier work on groupoids. For example, despite the importance of fibrations in [30], the earlier paper [3] on fibrations of groupoids by R Brown is not referenced. Brown's later paper [4] is referenced in [30], and contains the ideas of an action of a groupoid on a groupoid and of the semidirect product of groupoids.

In this thesis, we shall revisit some of these structural ideas, based on homotopy notions modelled by groupoids, but in the context of ordered groupoids. Our starting point is the paper [3] , in which fundamental constructions for fibrations of groupoids are explored, and applications given to the non-abelian cohomology of groups. The groupoid notions of the covering homotopy property and the path-lifting property for a map of groupoids $p : G \to H$ are shown to be equivalent, and equivalent to p being star-surjective. Any one of these properties thus defines p as a *fibration*. Steinberg [30] adopts star-surjectivity as the definition of a fibration of ordered groupoids. He undertakes a general study of factorization theorems for ordered groupoids, based on the semidirect product construction for one ordered groupoid acting on another and a left adjoint Der – the derived ordered groupoid – to the semidirect product. Steinberg uses the derived ordered groupoid to show that every ordered groupoid morphism factors in a canonical way through a fibration, and that a star-injective morphism has a universal factorization through a covering. The latter result is Ehresmann's Maximum Enlargement Theorem [12]. Lawson's approach to this result [21] also demonstrates its applications in inverse semigroup theory, and in particular to the theory of idempotent pure homomorphisms. Another important precursor to this thesis is the thesis of Matthews [26] and the paper [22]. Matthews studies inverse semigroups and ordered groupoids from the point of view of abstract homotopy theory, showing that the category of ordered groupoids admits cocylinders, and as an application is able to deduce Steinberg's Fibration Theorem.

In chapter 2, we study the category of ordered groupoids. Our main construction is the ordered quotient groupoid, depending on a new ordered version of the notion of normal subgroupoid and the construction of an ordered quotient groupoid. This construction gives a canonical factorization of any ordered functor of ordered groupoids as a star-surjective functor followed by a star-injective functor.

In chapter 3 we study fibrations of ordered groupoids: in the ordered case we find that being star-surjective and possessing the covering homotopy property are no longer equivalent properties. We therefore have *fibrations* – the star-surjective functors – and *strong fibrations* that have the covering homotopy property. We establish the basic properties of these classes of functors, following [3].

In chapter 4 we study the action of an ordered groupoid G on an ordered groupoid A and construct the semidirect product $G \ltimes A$, following [4]. Then we consider the special case when A is a poset, for which the categories of actions and coverings of ordered groupoids are equivalent. Further, we link our work on ordered quotients in chapter 2 with Ehresmann's Maximum Enlargement Theorem, which gives a factorization of any star-injective functor as a special kind of embedding – called an *enlargement* – followed by a covering. So we show that any ordered functor has a canonical factorization as an ordered fibration, followed by an ordered enlargement, followed by an ordered covering. We conclude this chapter by giving a direct and simplified approach to the main results of Steinberg in [30].

In chapter 5 we look at constructions with a more homological basis. Our main construction is the derived module D_θ, which has been considered for inverse semigroups in [14] and for groupoids in [9, 10]. We also involve D_θ in constructing adjoint functors between the category of ordered crossed complexes $\mathcal{OCRS}$ and the category of ordered chain complexes $\mathcal{OCHN}$.

Chapter 6, is about cohomology of crossed modules over groups. We describe the monoid of derivations $Der(G, A)$ over the crossed module μ. We also obtain a monoid structure on the cohomology set $H^1(G, A)$ from the monoid structure of the derivations monoid. Further, we study the derivations linked to endomorphisms of the crossed module μ and describe a monoid structure for that kind of derivations $Der_\diamond(G, A)$. The conclusion is to construct a new groupoid $Gpd_\diamond(\mu, \mu)$ with vertex set $End(A, G, \mu)$ and arrow set $Der_\diamond(G, A)$. This chapter extends some results for non-abelian cohomology derived in [3] using fibrations of groupoids.

Finally, chapter 7 considers the extension of some results of chapter 6 to ordered crossed modules. This uses the monoidal closed structure on the category of crossed complexes given in [8], and by combining results of [6] and [14] we obtain quick proofs of results of Brown and İçen in [11].

Some of the results in chapters 2, 3, and 4 have been collated as [1].

1.1 Groupoids

A *groupoid* G is a small category in which every arrow is invertible. We consider a groupoid as an algebraic structure, so elements are the arrows of G and composition is an associative partial binary operation. The set of identities in G is denoted $\mathrm{Ob}(G)$ and an element $g \in G$ has domain $d(g) = gg^{-1}$ and range $r(g) = g^{-1}g$.
A groupoid G is connected if $G(x, y)$ is non-empty for all objects x, y of G. The components of G are the maximal connected subgroupoids of G, denoted $\pi_0 G$. If every arrow in G is an identity, we say that G is trivial or discrete.

Example 1.1.1. The groupoid $\mathfrak{I}$ has two identities 0 and 1, and two non-identity arrows ι and ι^{-1} with $d(\iota) = 0$ and $r(\iota) = 1$.

Let $e \in \mathrm{Ob}(G)$. Then the vertex group at e is the set $G\{e\} = \{g \in G : d(g) = e = r(g)\}$, and the star of G at e (costar of G at e) is the set $\mathrm{star}_G(e) = \{g \in G : d(g) = e\}$ $(\mathrm{costar}_G(e) = \{g \in G : r(g) = e\})$. A functor $\phi : G \to H$ is said to be star-surjective if, for each $e \in \mathrm{Ob}(G)$, the restriction $\phi : \mathrm{star}_G(e) \to \mathrm{star}_H(e\phi)$ or $(\phi : \mathrm{costar}_G(e) \to \mathrm{costar}_H(e\phi))$ is surjective. Star-injective and star-bijective functors are defined similarly. A star-surjective functor is also called a fibration, a star-injective functor an immersion, and a star-bijective functor a covering.

Let $\phi : G \to H$ be a functor of groupoids. The fibre of ϕ over $x' \in Ob(H)$ is the inverse image $x'\phi^{-1}$ of x', that is $x'\phi^{-1} = \{g \in G : g\phi = x'\}$.
The kernel of a functor ϕ is the inverse image of the objects set $Ob(H)$, so it is the union of all fibres of ϕ over the set $Ob(H)$. That is $\ker\phi = \{g \in G : g\phi \in Ob(H)\}$.

Lemma 1.1.2. *([3, Proposition 1.2]) A functor $\phi : G \to H$ is star-injective if and only if its kernel is equal to* $\mathrm{Ob}(G)$.

Normal Subgroupoids and Quotient Groupoids

A subgroupoid N of the groupoid G is a subcategory that is itself a groupoid. We say a subgroupoid N is *wide* in G if N has the same objects as G. A subgroupoid N of G is called *full* if $N(x, y) = G(x, y)$ for all objects x, y of N.

A subgroupoid N of G is called *normal* if N is wide in G and if $g \in G$ and $n \in N$ with $g^{-1}ng$ defined in G, then $g^{-1}ng \in N$.

We remark that $g^{-1}ng$ is defined if and only if $n \in G\{d(g)\}$. Now N may well contain elements not in local groups, and this definition follows that given by Higgins in [18]. However, we note that Matthews [26] assumes that if $n \in G$ then $n \in G\{e\}$ for some $e \in \mathrm{Ob}(G)$. This distinction will be of importance in Chapter 3. A normal subgroupoid N then determines an equivalence relation $\simeq_N$ on G, defined by

$$g \simeq_N h \Longleftrightarrow \text{there exist } m, n \in N \text{ such that } g = mhn.$$

The set G/N of $\simeq_N$ equivalence classes inherits a natural groupoid structure, see [18]. The kernel of a functor $\phi : G \to H$ is a normal subgroupoid, and the canonical map $G \to G/\ker\phi$ is a fibration, and ϕ then induces an immersion $G/\ker\phi \to H$.

The category whose objects are groupoids and whose arrows are groupoid morphisms is called the category of groupoids, denoted $\mathcal{GPD}$.

Chapter 2

The Category of Ordered Groupoids

The concept of an *ordered groupoid* and some properties of the category of ordered groupoids $\mathcal{OG}$ are discussed in the first section. The second section investigates categorical properties in the category $\mathcal{OG}$ such as the *cartesian closed* property and the *exponential law*. The third section is dedicated to the construction of the *ordered quotient groupoid* depending on a new definition of normality for ordered subgroupoids, with supportive examples provided.

2.1 Ordered Groupoids

Definition An *ordered* groupoid G is a groupoid together with a partial ordering on the vertices and arrows that satisfies the following

OG1 If α and β are arrows in G with $\alpha \leqslant \beta$ then $\alpha^{-1} \leqslant \beta^{-1}$.

OG2 If α, β, γ and δ are arrows in G with $\alpha \leqslant \beta$ and $\gamma \leqslant \delta$ then $\alpha\gamma \leqslant \beta\delta$ whenever $\alpha\gamma$ and $\beta\delta$ are defined. In particular, if $\alpha \leqslant \beta$ then $\alpha\alpha^{-1} \leqslant \beta\beta^{-1}$ and so $d(\alpha) \leqslant d(\beta)$ and similarly $r(\alpha) \leqslant r(\beta)$.

OG3 If α is an arrow and e is a vertex with $e \leqslant d(\alpha)$ then there is a unique arrow, called the restriction of α at e, and denoted $(e|\alpha)$, such that $(e|\alpha) \leqslant \alpha$, and $d(e|\alpha) = e$.

An alternative to OG3 is

OG4 If α is an arrow and f is a vertex with $f \leqslant r(\alpha)$ then there is a unique arrow, called the corestriction of α at f, and denoted $(\alpha|f)$, such that $(\alpha|f) \leqslant \alpha$, and $r(\alpha|f) = f$.

Let G, H be ordered groupoids. A functor $p : G \to H$ is an ordered functor if for all $g_1, g_2 \in G$ with $g_1 \leqslant g_2$ we have $g_1 p \leqslant g_2 p$. Ordered groupoids with ordered functors form the category of ordered groupoids $\mathcal{OG}$.

Example 2.1.1. (i) Any groupoid, with equality partial order $a \leqslant b \Longleftrightarrow a = b$ is an ordered groupoid.

(ii) A group morphism $\theta : G \to H$ gives an ordered groupoid in which vertices are the identities e_G and e_H, arrows are the disjoint union of elements $G \sqcup H$. a, b are composable only if $a, b \in G$ or $a, b \in H$. The ordering relation is defined as $h \leqslant g \Longleftrightarrow h = g\theta$.

(iii) To generalize the previous example, let P any poset, for each $p \in P$ choose a group G_p and for all $p, q \in P$ with $p \geqslant q$ let $\theta_{pq} : G_p \to G_q$ be a group morphism that satisfies the conditions:

- θ_{pp} is the identity of G_p.
- If $p \geqslant q \geqslant r$, then $\theta_{pq}\theta_{qr} = \theta_{pr}$.

then the disjoint union $\sqcup G_p$ is an ordered groupoid with ordering relation defined as

$$h \leqslant g \Longleftrightarrow h = g\theta_{pq} \text{ , where } g \in G_p \text{ and } h \in G_q.$$

(iv) Any poset is a discrete ordered groupoid.

Definition The set of identities of an ordered groupoid is a poset, and if this poset is a *meet semilattice* then, the ordered groupoid is called *inductive*.

If an ordered functor between inductive groupoids preserves the *meet* then, it is said to be an inductive functor. Inductive groupoids with ordered functors form a subcategory of $\mathcal{OG}$, as well the category of inductive groupoids with inductive functors.
From each inductive groupoid, G, we can obtain a semigroup by defining an associative operation $*$ on elements of G (denoted by $\otimes$ in [21]) as follows

$$g * h = (g|z)(z|h)$$

where g, h are arrows in G and z is the greatest lower bound of $r(g)$ and $d(h)$. This special kind of semigroup is called an *inverse* semigroup. We can reverse the above process to obtain an inductive groupoid $G(S)$ from any inverse semigroup S. The arrows of $G(S)$ are the elements of S and $\mathrm{Ob}(G(S))$ is identified with the set of idempotents of S, with $d(s) = ss^{-1}$ and $r(s) = s^{-1}s$. This gives an isomorphism between the categories of inverse semigroups (and inverse semigroup homorphisms) and inductive groupoids (and meet-preserving ordered functors). Lawson [21] calls this the Ehresmann-Schein-Nambooripad Theorem.

Example 2.1.2. (i) Let $\theta : G \longrightarrow H$ be a group morphism which makes $G \sqcup H$ an ordered groupoid and define $*$ on elements of $G \sqcup H$ as follows :

- if $g_1, g_2 \in G$ then $g_1 * g_2 = g_1 g_2 \in G$,
- if $h_1, h_2 \in H$ then $h_1 * h_2 = h_1 h_2 \in H$,
- and if $g \in G, h \in H$ then $g * h = (g\theta)h \in H$ and $h * g = h(g\theta) \in H$.

$G \sqcup H$ is an inductive groupoid, and $*$ makes it into an inverse semigroup.

(ii) Let $\mathfrak{I}$ denote the groupoid with two objects $0, 1$ and only two nonidentity elements ι, ι^{-1}, and let $\{e\}$ be a trivial group such that $e \leqslant 0, e \leqslant 1$ then we have an inductive groupoid $\mathfrak{I} \sqcup \{e\}$.
Define $*$ using the restrictions definition, since the only element in $\{e\}$ is the identity so all restrictions give this identity and that turns $\mathfrak{I} \sqcup \{e\}$ into an inverse semigroup. This is the *Brandt* semigroup B_2.

Many properties are satisfied in the category $\mathcal{OG}$, one of them is proved in the following proposition.

Proposition 2.1.3. *The pullback exists in the category $\mathcal{OG}$.*

Proof. Consider the following square

$$\begin{tikzcd} Q \arrow[r, "i"] \arrow[d, "q"'] & G \arrow[d, "p"] \\ X \arrow[r, "f"'] & H \end{tikzcd}$$

where X, G, H are ordered groupoids and f, p are ordered functors. Let Q be the subset of the product $X \times G$ determined by the condition $Q = \{(x, g) : xf = gp\}$ and q, i be the projections to X and G respectively.

The groupoid structure of Q is to be done component wise, and to show that Q is ordered define first the ordering relation as $(x, g) \leqslant (y, h)$ in Q if and only if $x \leqslant y$ in X and $g \leqslant h$ in G. Then we have the ordered groupoid axioms

OG1 If $(x, g) \leqslant (y, h)$ then $x \leqslant y$ in X and $g \leqslant h$ in G and since X and G are ordered groupoids we have $x^{-1} \leqslant y^{-1}$ in X and $g^{-1} \leqslant h^{-1}$ in G which implies $(x^{-1}, g^{-1}) \leqslant (y^{-1}, h^{-1})$ that is $(x, g)^{-1} \leqslant (y, h)^{-1}$.

OG2 If $(x, g) \leqslant (y, h)$ and $(z, k) \leqslant (t, l)$ such that $(x, g)(z, k)$ and $(y, h)(t, l)$ are defined, then we have $xz \leqslant yt$ in X and $gk \leqslant hl$ in G so $(x, g)(z, k) \leqslant (y, h)(t, l)$.

OG3 If (e, u) is an identity in Q and (x, g) an element starts at (e, u), and if $(k, v) \leqslant (e, u)$ then the restrictions $(k|x)$ and $(v|g)$ satisfy the following
$(k|x)f \leqslant xf$, since f is ordered, and starts at kf. And $(v|g)p \leqslant gp$, since p is

ordered, and starts at vp.
But $kf = vp$ and $xf = gp$, hence by uniqueness of restriction in H we have $(k|x)f = (v|g)p$ and so $((k|x), (v|g)) \in Q$.

□

2.2 The Cartesian Closed Property and The Exponential Law in The Category $\mathcal{OG}$

In this section we explain the definition of a cartesian closed category and then we describe this structure in $\mathcal{OG}$ using the *internal hom* functor. In addition, we define the composition functor $\mu : \mathsf{OGPD}(G, H) \times \mathsf{OGPD}(H, K) \to \mathsf{OGPD}(G, K)$ and consider the special case $\mu : \mathsf{END}(G) \times \mathsf{END}(G) \to \mathsf{END}(G)$ to conclude that the *interchange law* does hold for the *endomorphism groupoid*, that is $\mathsf{END}(G)$ is a monoidal ordered groupoid.

Definition The category $\mathcal{C}$ is called cartesian closed (see [25, section IV.6]) if and only if it satisfies the following three properties:

- It has a terminal object.
- Any two objects X and Y of $\mathcal{C}$ have a product $X \times Y$ in $\mathcal{C}$.
- Any two objects Y and Z of $\mathcal{C}$ have an exponential Z^Y in $\mathcal{C}$.

The first two conditions can be combined to the single requirement that any finite ,possibly empty, family of objects of $\mathcal{C}$ admit a product in $\mathcal{C}$, because of the natural associativity of the categorical product and because the empty product in a category is the terminal object of that category. The third condition is equivalent to the requirement that the functor $- \times Y$, i.e. the functor from $\mathcal{C}$ to itself that maps objects X to $X \times Y$ and morphisms ϕ to $\phi \times \mathrm{id}_Y$, has a right adjoint, usually denoted $-^Y$, for all objects Y in $\mathcal{C}$. For small categories, this can be expressed by the existence of a natural bijection between the hom-sets

$$Hom(X \times Y, Z) \cong Hom(X, Z^Y).$$

The cartesian closed structure

It is well-known that the category of groupoids is cartesian closed: the cartesian closed structure is just the restriction of that on the category of categories [25]: see also [3] and [10, Appendix C]. There are number of ways of showing that a category is cartesian closed.

For the category $\mathcal{OG}$ of ordered groupoids, we shall first describe the *internal hom* functor. This is just the restriction of the internal hom in the category of groupoids to the ordered case. So given two ordered groupoids G, H the ordered groupoid $\mathsf{OGPD}(G, H)$ or H^G is defined as follows. The objects of $\mathsf{OGPD}(G, H)$ are the ordered morphisms $G \to H$ and an arrow from $f : G \to H$ to $g : G \to H$ is an ordered natural transformation, which is an ordered function $\tau : \mathrm{Ob}(G) \to H$ such that, for all arrows $a \in G(x, y)$ the following square

$$\begin{tikzcd} xf \arrow[r, "af"] \arrow[d, "x\tau"'] & yf \arrow[d, "y\tau"] \\ xg \arrow[r, "ag"'] & yg \end{tikzcd}$$

commutes in H. We write $\tau : f \implies g$. In this case, f and τ determine g since for any $a \in G$ we have $ag = (x\tau)^{-1}(af)(y\tau)$. Given ordered natural transformations $\tau : f \implies g$ and $\sigma : g \implies k$ their composition is the ordered natural transformation $\tau \cdot \sigma : f \implies k$ defined by $x(\tau \cdot \sigma) = (x\tau)(x\sigma)$ where $d(x\sigma) = xg = r(x\tau)$. This makes $\mathsf{OGPD}(G, H)$ a groupoid, since an ordered natural transformation τ has inverse $\bar{\tau} : x \mapsto (x\tau)^{-1}$.

Lemma 2.2.1. *If G, H are ordered groupoids then $\mathsf{OGPD}(G, H)$ is also an ordered groupoid.*

Proof. The groupoid structure has been described above. For the ordering on $\mathsf{OGPD}(G, H)$, suppose that $f, g : G \to H$ are ordered functors and that $f \leqslant g$, that is for all $a \in G$ we have $af \leqslant ag$. Suppose that $\sigma : g \implies h$, so that for all $x \in \mathrm{Ob}(G)$ we have $d(x\sigma) = xg$. Then $xf \leqslant xg$ and so $x\sigma$ has a unique restriction $(xf|x\sigma)$. This is an ordered function $\mathrm{Ob}(G) \to H$ and defines an ordered natural transformation from f. Moreover, suppose that $\tau : f \implies k$ and that $\tau \leqslant \sigma$, then for all $x \in \mathrm{Ob}(G)$ we have $d(x\tau) = xf$ and $x\tau \leqslant x\sigma$. Hence $x\tau = (xf|x\sigma)$ and so $\tau = (f|\sigma)$. $\square$

Remark 2.2.2. If $e, x \in \mathrm{Ob}(G)$ and $e \leqslant x$ then $ef \leqslant xf$ and $e\tau \leqslant x\tau$ with $d(e\tau) = ef$. Hence we have $e\tau = (ef|x\tau)$. If every object of G is below a maximal object, then τ is determined by its values on the maximal objects of $\mathrm{Ob}(G)$. In the special case that $\mathrm{Ob}(G)$ has a maximum m,then τ is determined by $m\tau$ and for all $x \in \mathrm{Ob}(G)$ we have $x\tau = (xf|m\tau)$.

We shall now identify an arrow in $\mathsf{OGPD}(G, H)$ with a pair (f, τ) where $f : G \to H$ is an ordered functor and $\tau : f \implies g$ is an ordered natural transformation. As already remarked, f and τ determine g. We now have an ordered functor $\varepsilon : G \times \mathsf{OGPD}(G, H) \to H$ given by

$$\varepsilon : (a, (f, \tau)) \mapsto (af)(r(a))\tau.$$

Lemma 2.2.3. *Given ordered groupoids G, H and K and an ordered functor $\gamma : G \times H \to K$, there exists a unique ordered functor $\lambda : H \to \mathsf{OGPD}(G, K)$ such that the diagram*

$$\begin{tikzcd} G \times H \arrow[d, "1_G \times \lambda"'] \arrow[dr, bend left, "\gamma"] & \\ G \times \mathsf{OGPD}(G,K) \arrow[r, "\varepsilon"'] & K \end{tikzcd}$$

commutes.

Proof. For $b \in H$ with $d(b) = p$ and $r(b) = q$, we define $p\lambda$ to be the ordered morphism $G \to K$ given by $a(p\lambda) = (a, p)\gamma$, and $b\lambda$ is the ordered natural transformation $p\lambda \implies q\lambda$ given by $x(b\lambda) = (x, b)\gamma$ for all $x \in \mathrm{Ob}(G)$. Hence if $d(a) = x$ and $r(a) = y$ we get a commutative square

$$\begin{tikzcd} \cdot \arrow[r, "(x,b)\gamma"] \arrow[d, "(a,p)\gamma"'] & \cdot \arrow[d, "(a,q)\gamma"] \\ \cdot \arrow[r, "(y,b)\gamma"'] & \cdot \end{tikzcd}$$

in K. Then

$$\begin{aligned} (a, b)(1_G, \lambda)\varepsilon &= (a, (p\lambda, b\lambda))\varepsilon \\ &= a(p\lambda)y(b\lambda) \\ &= (a, p)\gamma(y, b)\gamma \\ &= (a, b)\gamma \end{aligned}$$

□

The mapping $\nu : \gamma \mapsto \lambda$ defined in the lemma defines a function

$$\nu : Hom(G \times H, K) \to Hom(H, \mathsf{OGPD}(G, K)).$$

Now given any $\eta : H \to \mathsf{OGPD}(G, K)$ we can compose

$$1_G \times \eta : G \times H \to G \times \mathsf{OGPD}(G, K)$$

with ε to obtain $\delta : G \times H \to K$

$$\begin{tikzcd} G \times H \arrow[d, "1_G \times \eta"'] \arrow[dr, bend left, "\delta"] & \\ G \times \mathsf{OGPD}(G,K) \arrow[r, "\varepsilon"'] & K \end{tikzcd}$$

and the mapping $\eta \mapsto \delta$ is inverse to ν. Hence we have a natural bijection

$$\nu : Hom(G \times H, K) \to Hom(H, \mathsf{OGPD}(G, K)).$$

Proposition 2.2.4. *The bijection ν extends to an isomorphism of ordered groupoids*

$$\nu : \mathsf{OGPD}(G \times H, K) \to \mathsf{OGPD}(H, \mathsf{OGPD}(G, K)).$$

Proof. We have already defined ν on objects. On arrows ν arises from the bijection between functions $\mathrm{Ob}(G \times H) \to K$ and functions $\mathrm{Ob}(H) \to \mathbf{Set}(\mathrm{Ob}(G) \to K)$. $\square$

The Endomorphism Groupoid

The ordered functor

$$G \times \mathsf{OGPD}(G, H) \times \mathsf{OGPD}(H, K) \xrightarrow{\varepsilon \times 1} H \times \mathsf{OGPD}(H, K) \xrightarrow{\varepsilon} K$$

corresponds, under the isomorphism of Proposition 2.2.4, to an ordered functor

$$\mu : \mathsf{OGPD}(G, H) \times \mathsf{OGPD}(H, K) \to \mathsf{OGPD}(G, K)$$

called *composition*. On objects, this is just the composition of ordered functors, if $f : G \to H$ and $g : H \to K$ then $(f, g)\mu = fg$. Now given arrows (f, τ) and (g, σ) in $\mathsf{OGPD}(G, H)$ and $\mathsf{OGPD}(H, K)$ respectively, their composition $((f, \tau), (g, \sigma))\mu = (fg, \phi)$ where, for $x \in \mathrm{Ob}(G)$, we have $x\phi = (x\tau)g(r(x\tau))\sigma$. Of particular interest is the case when $G = H = K$. We then denote $\mathsf{OGPD}(G, G)$ by $\mathsf{END}(G)$; the functor $\mu : \mathsf{END}(G) \times \mathsf{END}(G) \to \mathsf{END}(G)$ then makes $\mathsf{END}(G)$ into a monoid in the category of ordered groupoids. In detail, we have

$$\mathsf{END}(G) = \{(f, \tau) : f \in End(G), \tau : \mathrm{Ob}(G) \to G, d(x\tau) = xf\}.$$

with the monoid operation given by

$$(f, \tau) \diamond (g, \sigma) = (fg, \tau g * \sigma),$$

where for $x \in \mathrm{Ob}(G)$,

$$x(\tau g * \sigma) = (x\tau)g(r(x\tau))\sigma.$$

The fact that this is a monoid in the category of groupoids implies that for any four arrows $(f, \tau), (g, \sigma), (h, \psi), (k, \phi) \in \mathsf{END}(G)$ with $(f, \tau)(g, \sigma)$ and $(h, \psi)(k, \phi)$ defined in the groupoid composition on $\mathsf{END}(G)$, we have the *interchange law* (see [25, section I.5])

$$((f,\tau)(g,\sigma)) \diamond ((h,\psi)(k,\phi)) = ((f,\tau) \diamond (h,\psi))((g,\sigma) \diamond (k,\phi)).$$

It is worth seeing why this works in the current setting. On the left hand side we have

$$\begin{aligned} ((f,\tau)(g,\sigma)) \diamond ((h,\psi)(k,\phi)) &= (f, \tau \cdot \sigma) \diamond (h, \psi \cdot \phi) \\ &= (fh, (\tau \cdot \sigma)h * (\psi \cdot \phi)) \\ &= (fh, (\tau h \cdot \sigma h) * (\psi \cdot \phi)). \end{aligned}$$

On the right hand side we have

$$\begin{aligned} ((f,\tau) \diamond (h,\psi))((g,\sigma) \diamond (k,\phi)) &= (fh, \tau h * \psi)(gk, \sigma k * \phi) \\ &= (fh, (\tau h * \psi).(\sigma k * \phi)). \end{aligned}$$

Since $\tau : f \implies g$ and $\psi : h \implies k$, it is easy to see that $\tau h * \psi : fh \implies gk$ and the composition here is defined. Now for $x \in \mathrm{Ob}(G)$,

$$x(\tau h \cdot \sigma h) * (\psi \cdot \phi) = (x\tau h)(x\sigma h)(r(x\sigma))\psi(r(x\sigma))\phi$$

whilst

$$x(\tau h * \psi) \cdot (\sigma k * \phi) = (x\tau h)(r(x\tau))\psi(x\sigma k)(r(x\sigma))\phi.$$

Because ψ is a natural transformation $h \implies k$ we have the following commutative square for the arrows $x\sigma h$ and $x\sigma k$

$$\begin{tikzcd} \cdot \arrow[r, "x\sigma h"] \arrow[d, "xg\psi"'] & \cdot \arrow[d, "(r(x\sigma))\psi"] \\ \cdot \arrow[r, "x\sigma k"'] & \cdot \end{tikzcd}$$

But here $xgh = r(x\tau h) = (r(x\tau))h$, and so $xg\psi = (r(x\tau))\psi$, so that

$$(x\sigma h)(r(x\sigma))\psi = (r(x\tau))\psi(x\sigma k)$$

and the interchange law does hold.

2.3 Quotients of Ordered Groupoids

The aim of this section is to establish the correct concept of the quotient of an ordered groupoid by an ordered normal subgroupoid. We mention two earlier approaches to this problem:

- If we use the unordered notion of a normal subgroupoid and the *Joubert* ordering (see [20]) on the quotient, this does not always result in an ordered groupoid, although it does in some cases, such as that arising from Matthew's definition of a normal subgroupoid [26]. The Joubert ordering is defined by $[a] \leqslant [b]$ if and only if, for each $b' \in [b]$ there exists $a' \in [a]$ with $a' \leqslant b'$.

- Lawson's notion of the kernel of an ordered groupoid morphism, as in [20], takes no account of the partial nature of the composition in a groupoid and requires the introduction of the concept of a *special* ordered functor.

In the unordered setting any groupoid morphism $\theta : G \to H$ factorises as the composition of a canonical fibration $G \to G/\ker\theta$ followed by a star-injective morphism $G/\ker\theta \to H$. However, if $\theta : G \to H$ is ordered, the quotient groupoid $G/\ker\theta$ may not admit an ordering so that we get an ordered fibration $G \to G/\ker\theta$. The following example clarifies this case

Example 2.3.1. Consider the simplicial groupoid $B = \mathbb{N} \times \mathbb{N}$ in which $\mathbb{N}$ carries the reverse of its usual total order. We order arrows in B by $(p+m, p+n) \leqslant (m, n)$ for all $p, m, n \in \mathbb{N}$, hence $(0,0)$ is a maximum element. As a poset, $\mathbb{N}$ is now an infinite descending chain, and so B is inductive. The inverse semigroup corresponding to B is the *bicyclic monoid*, see [21]. We define $\theta : B \to \mathbb{Z}_2$ by $(m, n) \mapsto m - n \pmod 2$, so that $\ker\theta = \{(m,n) : m - n \text{ is even }\}$. Then $B/\ker\theta$ is isomorphic to the groupoid $\mathfrak{I}$ which can only carry the trivial ordering and hence $B \to \mathfrak{I}$ is not an ordered functor.

We make a new definition of an ordered normal subgroupoid A of an ordered groupoid G and of the congruence relation that it imposes on G so as to make the ordered quotient $G/_{\leqslant} A$ an ordered groupoid in a natural way.

Definition A subgroupoid A of an ordered groupoid G is a *normal ordered subgroupoid* if:

NO1 A is *wide* in G: that is, $\mathrm{Ob}(A) = \mathrm{Ob}(G)$,

NO2 If $a \in A$ and $e \leqslant aa^{-1}$ then the restriction $(e|a) \in A$,

NO3 If $a \in A$ and $h, k \in G$ satisfy:

1. $h \leqslant g$ and $k \leqslant g$ for some $g \in G$, that is g is an upper bound for h and k,
2. $h^{-1}ak$ is defined in G,

then $h^{-1}ak \in A$.

An alternative to NO3 is the following condition :

NO4 If $g \in G$ and $a \in A$ with $r(a) \leqslant d(g)$ and $d(a) \leqslant d(g)$ then

$$(d(a)|g)^{-1}a(r(a)|g) \in A\,.$$

Remark 2.3.2.

- In the unordered case we recover Higgins' definition of a normal subgroupoid [18]. The second condition is vacuous, and in the third condition we have $h = k$ and then $h^{-1}ak$ being defined forces a to be in a local subgroup of G.
- In the ordered case, if A is a disjoint union of groups, then $h^{-1}ak$ being defined implies that $hh^{-1} = kk^{-1}$, and then each of h and k is the restriction of g, so by uniqueness of restriction, $h = k$ again and we recover Matthew's definition [26].

Lemma 2.3.3. *The kernel of an ordered morphism $\theta : G \to H$ is a normal ordered subgroupoid of G.*

Proof. NO1 We have θ is a morphism so it is mapping identities in G to identities in H so $\mathrm{Ob}(G) \subseteq \ker\theta$ meaning $\ker\theta$ is wide in G.

NO2 Let $a \in \ker\theta$ and $e \leqslant aa^{-1}$, then $(e|a)\theta = (e\theta|a\theta) \in H_0$ so $(e|a) \in \ker\theta$.

NO3 Suppose that $a \in \ker\theta$, $h, k \in G$ with the composition $h^{-1}ak$ defined in G, and that there exists $g \in G$ with $h \leqslant g$ and $k \leqslant g$. Let $a\theta = z \in H_0$. Then $(h^{-1}\theta)(k\theta)$ is defined in H, and $h\theta \leqslant g\theta$, $k\theta \leqslant g\theta$. It follows that $h\theta = (z|g\theta) = k\theta$ and hence $(h^{-1}ak)\theta = (h^{-1}\theta)(k\theta) = (h\theta)^{-1}(k\theta) \in H_0$ and so $h^{-1}ak \in \ker\theta$.

□

Example 2.3.4. For an inverse semigroup S, let $E(S)$ denote its set of idempotents. An inverse subsemigroup K of an inverse semigroup S is *self-conjugate* if, for all $s \in S$, $s^{-1}Ks \subseteq K$. We say K is *normal* if K is self-conjugate and if $E(K) = E(S)$ (see [21, section 5.1]).

Let S be an inverse semigroup, K an inverse subsemigroup of S and suppose that $G(K), G(S)$ are the correspond inductive groupoids of K and S respectively. Then K is a normal inverse subsemigroup of S if and only if $G(K)$ is a normal ordered subgroupoid of $G(S)$. Suppose that K is normal. We show that $G(K)$ is a normal ordered subgroupoid of $G(S)$.

NO1 $\mathrm{Ob}(G(K)) = \mathrm{Ob}(G(S))$ from the assumption that K is normal.

NO2 If $k \in G(K)$ and $e \leqslant kk^{-1}$, then $(e|k) \leqslant k$ and $(e|k) = ek$ in K so $(e|k) \in K$ and $(e|k) \in G(K)$.

NO3 Consider a composition $s^{-1}kt$ defined in $G(S)$ where $s \leqslant u$ and $t \leqslant u$ for some $u \in G(S)$. Then we have in S that $s^{-1}kt \leqslant u^{-1}ku$ but we know that $u^{-1}ku \in K$, so the composed arrow $u^{-1}ku$ is in $G(K)$ and so is $s^{-1}kt$ since $s^{-1}kt \leqslant u^{-1}ku$.

Now, suppose that $G(K)$ is a normal ordered subgroupoid of $G(S)$. Then

$$E(K) = \mathrm{Ob}(G(K)) = \mathrm{Ob}(G(S)) = E(S)$$

and so we need in addition that K is a self-conjugate inverse subsemigroup of S. We have $s^{-1}kt \in G(K)$ whenever $s^{-1}kt$ is defined in $G(S)$ and there is an upper bound of s and t in $G(S)$. Consider $s^{-1}ks$ with $k \in K$ and $s \in S$. We can write, in S:

$$s^{-1}ks = (s^{-1}kss^{-1}k^{-1})(ss^{-1}kss^{-1})(k^{-1}ss^{-1}ks)$$

where the bracketed terms on the right-hand side are composable in $G(S)$. And $ss^{-1}kss^{-1} \leqslant k$ so $ss^{-1}kss^{-1} \in G(K)$. Moreover we have $s^{-1}kss^{-1}k^{-1} \leqslant s^{-1}$ and $k^{-1}ss^{-1}ks \leqslant s$ so the product of the three terms is in $G(K)$ and so, as an element of S, is in K.

Trying to solve the first problem mentioned at the beggining of this section, we use the technique of constructing a poset as a quotient of a preordered set. So we define an equivalence relation on G that leads to a partial ordering relation on the quotient $G/_{\leqslant} A$ where we assume that A is a normal ordered subgroupoid of the ordered groupoid G.

Lemma 2.3.5. *Let A be a normal ordered subgroupoid of the ordered groupoid G. Then the relation*

$$g \simeq_A h \iff \textit{there exist } a, b, c, d \in A \textit{ such that } agb \leqslant h \textit{ and } chd \leqslant g$$

is an equivalence relation on G. And the relation

$$[g] \leqslant [k] \Leftrightarrow \textit{there exist } a, b \in A \textit{ such that } agb \leqslant k$$

is a well-defined partial order on the set $G/_{\leqslant} A$ of equivalence classes of $\simeq_A$.

Proof. (i) $\simeq_A$ is reflexive since $d(g)gr(g) \leqslant g$, and $d(g), r(g) \in A$.

(ii) It is clear from the definition of $\simeq_A$ that it is symmetric.

(iii) Suppose that $g \simeq_A h$ and $h \simeq_A k$. Then there exist $a, b, u, v \in A$ such that $agb \leqslant h$ and $uhv \leqslant k$. Then $aa^{-1} \leqslant hh^{-1} = u^{-1}u$ and $b^{-1}b \leqslant h^{-1}h = vv^{-1}$, and $(u|aa^{-1})agb(b^{-1}b|v) \leqslant uhv \leqslant k$ with $(u|aa^{-1})a, b(b^{-1}b|v) \in A$. Similarly there exist $p, q \in A$ with $pkq \leqslant g$ and $g \simeq_A k$.

We now show that $\leqslant$ is well-defined. Suppose that $g \simeq_A g'$ and $k \simeq_A k'$. Hence there exist $p, q, u, v \in A$ with $pg'q \leqslant g$ and $ukv \leqslant k'$. Now if in addition there exist $a, b \in A$ with $agb \leqslant k$ then we have

$$(u|aa^{-1})agb(b^{-1}b|v) \leqslant ukv \leqslant k'$$

and

$$((u|aa^{-1})a|pp^{-1})pg'q(q^{-1}q|b(b^{-1}b|v)) \leqslant (u|aa^{-1})agb(b^{-1}b|v).$$

Now it is clear that

(i) $\leqslant$ is reflexive since $d(g)gr(g) \leqslant g$ makes $[g] \leqslant [g]$.

(ii) Transitive for if $[g] \leqslant [h]$ and $[h] \leqslant [k]$ then there are $a, b, c, d \in A$ such that $agb \leqslant h$ and $chd \leqslant k$ and we have $(c|aa^{-1})agb(b^{-1}b|d) \leqslant k$ making $[g] \leqslant [k]$.

(iii) Antisymmetric directly from the definition of $\simeq_A$.

$\square$

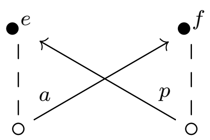
When we represent relationships in G diagrammatically, we shall indicate the partial order by a dashed line with the order decreasing down the page.

We now have a conception about the equivalence classes of arrows of G under $\simeq_A$, and we need to understand the classes of vertices of G under $\simeq_A$ in order to describe the ordered groupoid structure on $G/_{\leqslant} A$ eventually.

Lemma 2.3.6. *For identities of G, the relation $\simeq_A$ reduces to the following: if $e, f \in \mathrm{Ob}(G)$ then $e \simeq f$ if and only if there exist $a, p \in A$ such that*

$$aa^{-1} \leqslant e, a^{-1}a = f \textit{ and } pp^{-1} \leqslant f, p^{-1}p = e.$$

A pair (a, p) of arrows of A realizing the relation $\simeq_A$ for $e, f \in \mathrm{Ob}(G)$ is called an A-nexus between e and f.



Remark 2.3.7. The relation $\simeq_A$ is realized on $\mathrm{Ob}(G)$ by the action of a left cancellative category $C(A)$, discussed in chapter 3, on $\mathrm{Ob}(G) = \mathrm{Ob}(A)$ as

$$e \triangleleft (e, a) = f \text{ and } f \triangleleft (f, p) = e.$$

There is a natural ordered groupoid structure on $G/_{\leqslant} A$ that we now begin to describe.

Lemma 2.3.8. *Suppose that $g \simeq_A h$. Then $g^{-1} \simeq_A h^{-1}$ and $gg^{-1} \simeq_A hh^{-1}$.*

Proof. Let $a, b, u, v \in A$ such that $agb \leqslant h$ and $uhv \leqslant g$. Then

$$b^{-1}g^{-1}a^{-1} \leqslant h^{-1} \text{ and } v^{-1}h^{-1}u^{-1} \leqslant g^{-1}.$$

Furthermore,

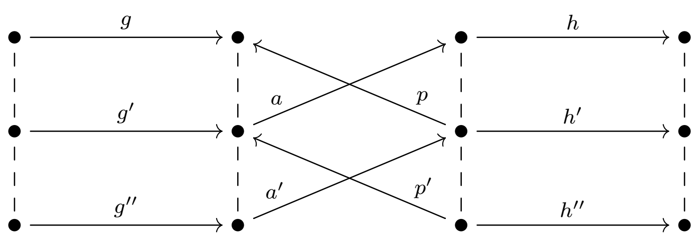
$$hh^{-1} \geqslant agbb^{-1}g^{-1}a^{-1} = agbb^{-1}g^{-1}gg^{-1}a^{-1}$$

with $agbb^{-1}g^{-1} \in A$. Similarly $gg^{-1} \geqslant uhvv^{-1}h^{-1}hh^{-1}u^{-1}$ with $uhvv^{-1}h^{-1} \in A$, and so $gg^{-1} \simeq_A hh^{-1}$. $\square$

Before we define a partial composition on the classes of $G/_{\leqslant} A$, we need the following lemma

Lemma 2.3.9. *Let $g, h \in G$ with $g^{-1}g \simeq_A hh^{-1}$. Let (a, p) be an A-nexus between $g^{-1}g$ and hh^{-1}. Define*

$$\begin{aligned} g' &= (g|aa^{-1}) \\ h' &= (pp^{-1}|h) \\ a' &= (a|pp^{-1}) \\ p' &= (p|aa^{-1}) \\ g'' &= (g|a'a'^{-1}) \\ h'' &= (p'p'^{-1}|h). \end{aligned}$$



Then $g'ah \simeq gp^{-1}h'$, and if we make an alternative choice of nexus (a_1, p_1) leading to elements g'_1 and h'_1, then

$$g'_1a_1h \simeq g'ah \simeq gp^{-1}h' \simeq gp_1^{-1}h'_1.$$

Proof. We have $g'ah \geqslant g''a'h' = g''a'pg^{-1}gp^{-1}h'$ with $g''a'pg^{-1} \in A$ by normality of A, and similarly $gp^{-1}h' \geqslant g'p'^{-1}h'' = g'ahh^{-1}a^{-1}p'^{-1}h''$ with $h^{-1}a^{-1}p'^{-1}h'' \in A$.
For the alternative choice of nexus we have $a^{-1}a = a_1^{-1}a_1 = hh^{-1}$ and

$$g'_1a_1h \simeq_A g'_1a_1a^{-1}g'^{-1}g'ah$$

with $g_1'a_1a^{-1}g'^{-1} \in A$. Hence $g_1'a_1h \simeq_A g'ah$ and similarly $gp^{-1}h' \simeq_A gp_1^{-1}h_1'$. $\square$

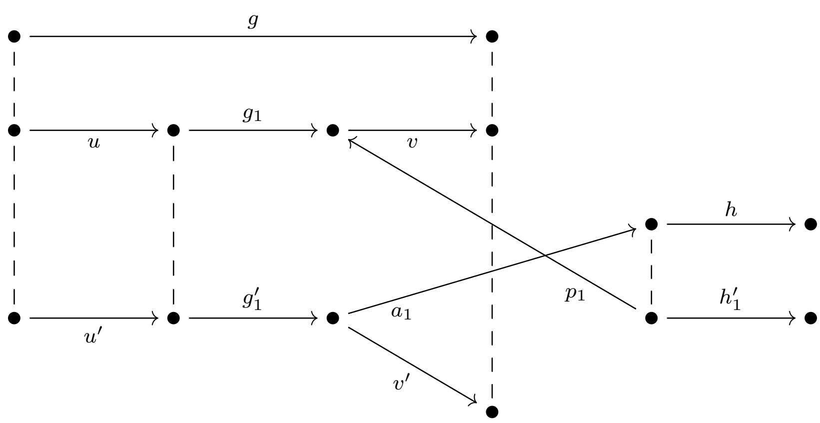
For given $g, h \in G$ with $g^{-1}g \simeq_A hh^{-1}$ we have defined a unique class in $G/_\leqslant A$ in terms of an A-nexus between $g^{-1}g$ and hh^{-1}, but independent of the A-nexus chosen. We denote this class temporarily by $g \between h \in G/_\leqslant A$. Hence in the notation of the lemma above

$$g \between h = [g'ah] = [gp^{-1}h'].$$

In the following lemma we show that $\between$ is well-defined in $G/_\leqslant A$.

Lemma 2.3.10. *Suppose that $g \simeq_A g_1$ and that $h \simeq_A h_1$. Then $g \between h = g_1 \between h_1$ in $G/_\leqslant A$.*

Proof. Fix h and choose an A-nexus (a_1, p_1) between $g_1^{-1}g_1$ and hh^{-1}. Let $g_1' = (g_1|a_1a_1^{-1})$ and $h_1' = (p_1p_1^{-1}|h)$. Then $g_1 \between h = [g_1'a_1h] = [g_1p_1^{-1}h_1'] \in G/_\leqslant A$. There exist $u, v \in A$ with $ug_1v \leqslant g$. Let $u' = (u|g_1'g_1'^{-1})$ and $v' = (a_1a_1^{-1}|v)$.



Now $v'^{-1}a_1$ is a component of an A-nexus between $g^{-1}g$ and hh^{-1} and so can be used to define $g \between h$. By uniqueness of restriction we have $(g|v'^{-1}v') = u'g_1'v'$ and so

$$\begin{aligned} g \between h &= [u'g_1'v'v'^{-1}a_1h] \\ &= [u'g_1'a_1h] \\ &= [g_1'a_1h] = g_1 \between h. \end{aligned}$$

Similarly $g_1 \between h = g_1 \between h_1$. $\square$

We now have a well-defined composition of $\simeq_A$ classes as $[g][h] = g \between h$. This composition is associative since: if $g^{-1}g \simeq_A hh^{-1}$ and $h^{-1}h \simeq_A kk^{-1}$, choose A-nexuses (a, p) between $g^{-1}g$ and hh^{-1} and (b, q) between $h^{-1}h$ and kk^{-1}. Then $g \between h = [g'ah]$ and $g'ah \between k = [g'ahq^{-1}k']$. But $h \between k = [hq^{-1}k']$ and $g \between hq^{-1}k' = [g'ahq^{-1}k']$.
This last observation then establishes the following result.

Proposition 2.3.11. *$G/_\leqslant A$ is a groupoid under the operation of composition of $\simeq_A$ classes.*

We shall now proceed to verify that the partial order on $\simeq_A$ classes given in Lemma 2.3.5 makes $G/_{\leqslant} A$ into an ordered groupoid.

Lemma 2.3.12. *If* $[g] \leqslant [h]$ *in* $G/_{\leqslant} A$ *then* $[g]^{-1} = [g^{-1}] \leqslant [h^{-1}] = [h]^{-1}$.

Proof. We have $agb \leqslant h$ for some $a, b \in A$ and so $b^{-1}g^{-1}a^{-1} \leqslant h^{-1}$. □

Lemma 2.3.13. *Suppose that* $[g_1] \leqslant [h_1]$ *and that* $[g_2] \leqslant [h_2]$ *in* $G/_{\leqslant} A$ *and that the compositions* $[g_1][g_2]$ *and* $[h_1][h_2]$ *exist. Then* $[g_1][g_2] \leqslant [h_1][h_2]$.

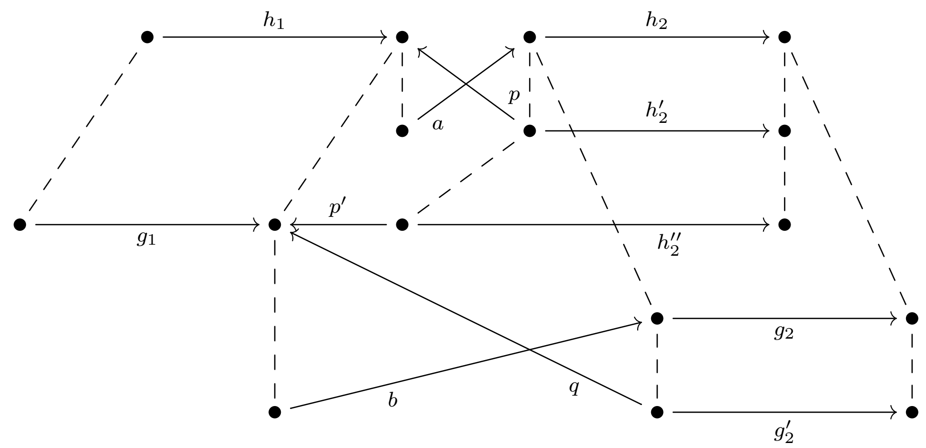
Proof. There exist $a_1, b_1, a_2, b_2 \in A$ such that $a_1g_1b_1 \leqslant h_1$ and $a_2g_2b_2 \leqslant h_2$. Since $[g_i] = [a_ig_ib_i]$ we may as well assume that $g_i \leqslant h_i$ for $i = 1, 2$. We have A-nexuses (a, p) between $h_1^{-1}h_1$ and $h_2h_2^{-1}$ and (b, q) between $g_1^{-1}g_1$ and $g_2g_2^{-1}$. We set $p' = (p|g_1^{-1}g_1)$ and $h_2'' = (p'p'^{-1}|h_2')$. Then

$$g_1q^{-1}g_2' \simeq_A g_1q^{-1}g_2'g_2'^{-1}qp'^{-1}h_2''$$

since $g_2', h_2'' \leqslant h_2$ and so $g_2'^{-1}qp'^{-1}h_2'' \in A$

$$g_1p'^{-1}h_2'' \leqslant h_1p^{-1}h_2'.$$

and so $[g_1][g_2] = [g_1q^{-1}g_2'] \leqslant [h_1p^{-1}h_2'] = [h_1][h_2]$.



□

Lemma 2.3.14. *Suppose that* $[e] \leqslant [gg^{-1}]$. *Then there exists a unique arrow* $[k] \in G/_{\leqslant} A$ *such that* $[k] \leqslant [g]$ *and* $[k][k]^{-1} = [e]$.

Proof. Since $[e] \leqslant [gg^{-1}]$ there exists $a \in A$ with $aa^{-1} \leqslant gg^{-1}$ and $a^{-1}a = e$. Let $g' = (aa^{-1}|g)$. Then $[g'] \leqslant [g]$ and $[g'][g']^{-1} = [aa^{-1}] = [e]$.
Now suppose that $[k] \leqslant [g]$ and $[k][k]^{-1} = [e]$. We shall show that $[k] = [g']$. There exist $u, v \in A$ such that $ukv \leqslant g$ and an A-nexus (b, q) between kk^{-1} and e so that

$$bb^{-1} \leqslant kk^{-1}, b^{-1}b = e, qq^{-1} \leqslant e, q^{-1}q = kk^{-1}.$$

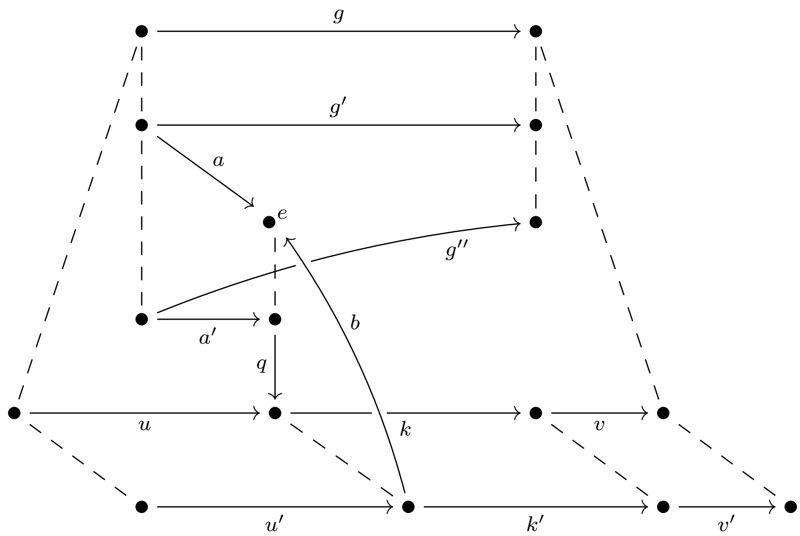
Let $k' = (bb^{-1}|k)$, $u' = (u|bb^{-1})$ and $v' = (k'^{-1}k'|v)$. Then $u'k'v' \leqslant ukv \leqslant g$ and

$$g' \simeq_A g'g'^{-1}ab^{-1}u'^{-1}u'k'v'$$

since $g' \leqslant g$, $u'k'v' \leqslant g$, and $ab^{-1}u'^{-1} \in A$,

$$g'g'^{-1}ab^{-1}u'^{-1}u'k'v' = ab^{-1}k'v' \simeq_A k' \leqslant k.$$

so we have $[g'] \leqslant [k]$
Let $a' = (a|qq^{-1})$ and $g'' = (a'a'^{-1}|g')$.



Then

$$k \simeq_A kvv^{-1}k^{-1}u^{-1}uq^{-1}a'^{-1}g''$$

since $ukv \leqslant g$, $g'' \leqslant g$ and $v, uq^{-1}a'^{-1} \in A$,

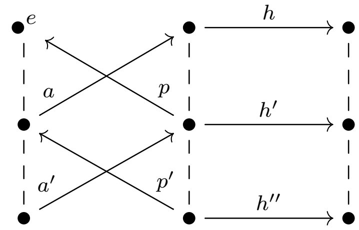
$$kvv^{-1}k^{-1}u^{-1}uq^{-1}a'^{-1}g'' = q^{-1}a'^{-1}g'' \simeq_A g'' \leqslant g'.$$

Hence $[k] \leqslant [g']$, and we have $[g'] \leqslant [k]$ so $[g'] = [k]$. $\square$

Theorem 2.3.15. *$G/_{\leqslant} A$ is an ordered groupoid and the quotient map $\varpi : G \to G/_{\leqslant} A$ is an ordered fibration.*

Proof. The ordered groupoid structure follows from Proposition 2.3.11 and Lemmas 2.3.12, 2.3.13, and 2.3.14. To show that $\varpi : g \mapsto [g]$ is a fibration, consider the restriction $\mathrm{star}_G(e) \to \mathrm{star}_{G/_{\leqslant} A}[e]$ and $[h] \in \mathrm{star}_{G/_{\leqslant} A}[e]$. There exists an A-nexus (a, p) between

e and hh^{-1}. Defining h', h'', a', p' as in Lemma 2.3.9,



we have $[p^{-1}h'] = [h']$ and $[h'] \leqslant [h]$. But

$$h \simeq_A hh^{-1}a^{-1}p'^{-1}h'' = a^{-1}p'^{-1}h'' \simeq_A h'' \leqslant h'$$

and so $[h] \leqslant [h']$. Hence we have $p^{-1}h' \in \mathrm{star}_G(e)$ and

$$\varpi : p^{-1}h' \mapsto [p^{-1}h'] = [h'] = [h].$$

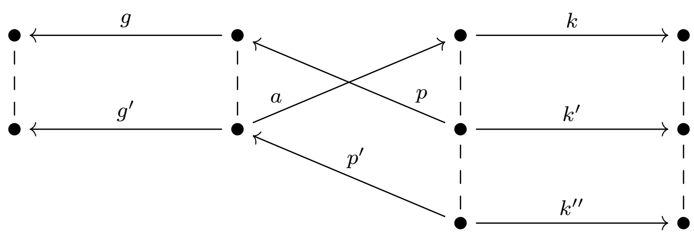
$\square$

If we now begin with an ordered functor $\theta : G \to H$, then by Lemma 2.3.3, its kernel $\ker\theta$ is a normal ordered subgroupoid of G and we can form the quotient ordered groupoid $G/_{\leqslant} \ker\theta$.

In the following theorem we show that θ factors through $G/_{\leqslant} \ker\theta$ as the composition of the fibration ϖ and an immersion.

Theorem 2.3.16. *An ordered functor $\theta : G \to H$ induces an immersion $\psi : G/_{\leqslant} \ker\theta \to H$ such that $\theta = \varpi\psi$.*

Proof. We define $\psi : G/_{\leqslant} \ker\theta \to H$ by $\psi : [g] \mapsto g\theta$. It is clear that ψ is an ordered functor. Suppose that $[k] \in \mathrm{star}_{G/_{\leqslant} \ker\theta}[gg^{-1}]$ and that $k\theta = g\theta$. Let (a, p) be a $\ker\theta$-nexus betwen gg^{-1} and kk^{-1}.



Now

$$(g^{-1}p^{-1}k')\theta = (g\theta)^{-1}(p\theta)^{-1}k'\theta = (g\theta)^{-1}k'\theta \leqslant (g\theta)^{-1}k\theta \in H_0.$$

It follows that $g^{-1}p^{-1}k' \in \ker\theta$, and so

$$k' = pg(g^{-1}p^{-1}k') \simeq_{\ker\theta} g.$$

But, as in the proof of Theorem 2.3.15 we have $[k'] \leqslant [k]$ and also

$$k \simeq_{\ker\theta} kk^{-1}a^{-1}p'^{-1}k'' \simeq_{\ker\theta} a^{-1}p'^{-1}k'' \simeq_{\ker\theta} k''.$$

Hence $[k] = [k''] \leqslant [k']$ and so $[k] = [k'] = [g]$. $\square$

Corollary 2.3.17. *If $\theta : G \to H$ is an ordered fibration then $\psi : G/_{\leqslant} \ker\theta \to H$ is an ordered covering.*

Proof. In the triangle

$$\begin{tikzcd}
\mathrm{star}_G(e) \arrow[r, "\varpi"] \arrow[d, "\theta"'] & \mathrm{star}_{G/_{\leqslant}\ker\theta}[e] \arrow[dl, "\psi"] \\
\mathrm{star}_H(e\theta) &
\end{tikzcd}$$

ψ is star-injective by Theorem 2.3.16, whilst θ and ϖ are star-surjective by assumption and by Theorem 2.3.15 respectively. It follows that ψ is star-surjective, and so ψ is a covering. $\square$

Example 2.3.18. For a groupoid G, the quotient G/G is isomorphic to the set $\pi_0 G$ of connected components of G, regarded as a trivial groupoid. If G is ordered, then its ordering induces a preorder on $\pi_0 G$, defined as follows:
If $\bar{g}$ denotes the connected component of $g \in G$, then $\bar{g} \leqslant \bar{h}$ if and only if for each $h' \in \bar{h}$ there exists $g' \in \bar{g}$ with $g' \leqslant h'$. As a preordered set, $\pi_0 G$ has a canonical partially ordered quotient $Q(G)$. To construct $Q(G)$ define an equivalence relation on $\pi_0 G$ as follows:

$$\bar{g} \simeq \bar{h} \Longleftrightarrow \bar{g} \leqslant \bar{h} \text{ and } \bar{h} \leqslant \bar{g}$$

The set $Q(G)$ of equivalence classes $[\bar{g}]$ is now a poset with partial order relation defined as

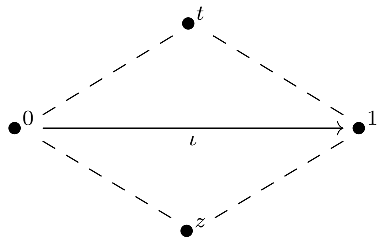
$$[\bar{g}] \leqslant [\bar{h}] \Longleftrightarrow \bar{g} \leqslant \bar{h}.$$

and the poset $Q(G)$ is isomorphic to the ordered quotient $G/_{\leqslant} G$.

We now provide some examples to illustrate this construction by starting with a nice ordered functor $\theta : G \to H$ and then an explicit description of the quotient $G/_{\leqslant} \ker\theta$.

Example 2.3.19. Let Δ^n be the simplicial groupoid on the set $I_n = \{0, 1, ..., n\}$ where $(\Delta^n)_0 = I_n$ and there is a unique arrow (i, j) from i to j. Δ^n is freely generated by the obvious directed chain Γ_n with vertex set I_n. We set $\Delta = \bigsqcup_{n \geqslant 1} \Delta^n$. The natural partial order is trivial between elements of each Δ^n. If $\gamma \in \Delta^p$ and $\delta \in \Delta^q$ with $p \leqslant q$ then $\gamma \geqslant \delta$ if and only if δ is the image of γ under an embedding $\Gamma_p \hookrightarrow \Gamma_q$. Δ is the ordered groupoid equivalent to the free inverse monoid on x denoted by $\mathcal{FIM}(x)$. This description of $\mathcal{FIM}(x)$ is essentially the same as that in [21, Theorem 9.5.4] as the Kachel semigroup

K_1.
Now let $G = \Delta$, and adjoin a minimum idempotent z and a maximum idempotent t to $\mathfrak{I}$ to form H.



We can then define an ordered functor $\theta : G \to H$ by mapping

$$\begin{aligned}\theta :&\Delta^0 \mapsto t\\ &\Delta^1 \mapsto \iota\\ &\Delta^p \mapsto z \text{ for all } p > 1\end{aligned}$$

Then $\ker\theta = \Delta^0 \cup \{0,1\} \cup \bigsqcup_{p>1} \Delta^p$. And $\Delta/_{\leqslant} \ker\theta = [\Delta^0] \cup \Delta^1 \cup [\Delta^p]$ where $[\Delta^p] = \{e_p : p \geqslant 2\}$ and e_p form an infinite descending chain.

The next example is about the *Bruck-Reilly extension* (see [29]) so we introduce it briefly as follows: Fix a group G and any endomorphism $\alpha : G \to G$. The *Bruck-Reilly extension*, $BR(G,\alpha)$ is the ordered groupoid with objects set $\{0,1,2,...\}$ and arrows of the form (m,g,n) where $m,n \geqslant 0$ and $g \in G$ with $d(m,g,n) = m$, $r(m,g,n) = n$ and the composition is defined as $(m,g,n)(n,h,k) = (m,gh,k)$ so $(m,g,n)^{-1} = (n,g^{-1},m)$. We get a groupoid $\mathbb{N} \times G \times \mathbb{N}$ with identities (m,e_G,m). The ordering is defined as follows:

$$(m,g,n) \geqslant (p,h,q) \text{ if and only if } m \leqslant p \text{ in } \mathbb{N},\ m-n = p-q \text{ and } h = g\alpha^{p-m}.$$

Example 2.3.20. Consider $\theta : BR(G,\alpha) \to BR(H,\beta)$. If we assume θ is the identity on objects, then it is determined by a group homomorphism $\bar{\theta} : G \to H$ such that $\theta : (m,g,n) \mapsto (m,g\bar{\theta},n)$.
Any such mapping is a functor but not necessary ordered, because for $(m,g,n) \geqslant (m+1, g\alpha, n+1)$ we have $(m,g\bar{\theta},n) \geqslant (m+1,g\alpha\bar{\theta},n+1)$ if and only if $g\bar{\theta}\beta = g\alpha\bar{\theta}$. Hence θ is an ordered functor if and only if $\bar{\theta}\beta = \alpha\bar{\theta}$, that is the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\bar{\theta}"'] & G \arrow[d, "\bar{\theta}"] \\ H \arrow[r, "\beta"'] & H \end{tikzcd}$$

Now we have $\ker\theta = \{(m,g,n) : m = n, g \in \ker\bar{\theta}\}$, and if $x \in \ker\bar{\theta}$ then $x\alpha\bar{\theta} = x\bar{\theta}\beta = e_H$ and $x\alpha \in \ker\bar{\theta}$.
To know the structure of $BR(G,\alpha)/_{\leqslant} \ker\theta$, let $(m,g_1,n) \simeq_{\ker\theta} (p,g_2,q)$ which means

there exist $(m, k_1, m), (n, k_2, n), (p, k_3, p), (q, k_4, q) \in \ker\theta$ such that $(m, k_1g_1k_2, n) \leqslant (p, g_2, q)$ and $(p, k_3g_2k_4, q) \leqslant (m, g_1, n)$ and from these two inequalities we have $m = p$ and $n = q$ and $k_1g_1k_2 \leqslant g_2$ and $k_3g_2k_4 \leqslant g_1$ in G so we can say that $g_1. \ker\bar{\theta}$ and $g_2. \ker\bar{\theta}$ are the same coset in $G/\ker\bar{\theta}$, and we propose that

$$BR(G, \alpha)/_{\leqslant} \ker\theta = BR(G/\ker\bar{\theta}, \bar{\alpha})$$

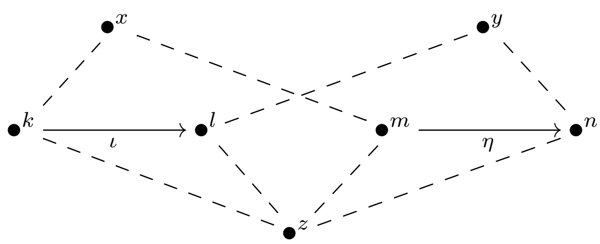
In such a case $\bar{\alpha}$ must be defined on $G/\ker\bar{\theta}$ by $(g.\ker\bar{\theta})\bar{\alpha} = g\alpha$ which is well-defined only if $\ker\bar{\theta} \subseteq \ker\alpha$.

Let us now consider the case in which $\bar{\theta} : G \to H$ is trivial, mapping $g \mapsto e_H$ for all $g \in G$ and $\bar{\theta}$ satisfies the last condition we have obtained $\ker\bar{\theta} \subseteq \ker\alpha$ for some $\alpha : G \to G$. Then by Theorem 2.3.16 we have the following commutative diagram

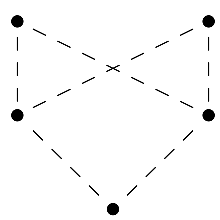
$$\begin{tikzcd}
BR(G,\alpha) \arrow[r, "\varpi"] \arrow[d, "\theta"'] & BR(\{e_G\}, \bar{\alpha}) \arrow[dl, "\psi"] \\
BR(H,\beta) &
\end{tikzcd}$$

where ψ is an isomorphism to $Im(\theta)$.

Example 2.3.21. If A is a normal inverse subsemigroup of an inverse semigroup S then $S/_{\leqslant} A$ need not be an inductive groupoid. Consider S as shown below:



and let $A = S$. The ordered groupoid $S/_{\leqslant} S$ is the trivial ordered groupoid



which is not inductive.

Chapter 3

Fibrations of Ordered Groupoids

We study the *covering homotopy property* for ordered fibrations and look at some differences from that in the unordered case in [3], and we support our conclusions by examples. We prove in the second section that a pullback of an ordered strong fibration is an ordered strong fibration. The third section studies two morphisms $Xp : \mathsf{OGPD}(X,G) \to \mathsf{OGPD}(X,H)$ and $iH : \mathsf{OGPD}(G,H) \to \mathsf{OGPD}(A,H)$ induced by the ordered fibration $p : G \to H$ and the ordered inclusion $i : A \to G$ respectively. Such morphisms in the unordered case are fibrations as in [3], but this is not the case for ordered fibrations as shown by examples.

3.1 Fibrations and Covering Homotopy Property in Ordered Groupoids

A groupoid morphism $p : G \to H$ is a fibration if it is star surjective. In [3] Brown shows that the possession of the *covering homotopy property* by p is equivalent to p being star surjective. That is for a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & G \arrow[d, "p"] \\ X \times \mathfrak{I} \arrow[r, "F"'] & H \end{tikzcd}$$

there exists a homotopy of groupoids $\widetilde{F} : X \times \mathfrak{I} \to G$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & G \arrow[d, "p"] \\ X \times \mathfrak{I} \arrow[r, "F"'] \arrow[ur, "\widetilde{F}"'] & H \end{tikzcd}$$

commute. For ordered groupoids a fibration is a star surjective functor between ordered groupoids. The posed question is that is it still true, as for the unordered case above, that the covering homotopy property is equivalent to star surjectivity in ordered groupoids? Part of the answer to this question is given in the following proposition.

Proposition 3.1.1. *Let E, G and H be ordered groupoids, i_0 be the inclusion map and f, p, F are ordered functors, such that the following diagram commutes*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "i_0"'] & G \arrow[d, "p"] \\ E \times \mathfrak{I} \arrow[r, "F"'] & H \end{tikzcd}$$

Then the covering homotopy property implies that p is an ordered fibration.

Proof. The *path lifting property* is the special case of the *covering homotopy property* in which the groupoid E is trivial groupoid with one identity. So suppose that $p : G \to H$ has the path lifting property. Let x be an identity of G and suppose that $h \in \mathrm{star}_H(xp)$. Let 0 denote the trivial groupoid with a single identity. Any morphism $0 \times \mathfrak{I} \to H$ is determined by the choice of a single arrow in H, and will be ordered. Thus x determines a morphism $0 \to G$ and h determines a morphism $0 \times \mathfrak{I} \to H$ making the commutative square

$$\begin{tikzcd} 0 \arrow[r, "f"] \arrow[d, "i_0"'] & G \arrow[d, "p"] \\ 0 \times \mathfrak{I} \arrow[r, "F"'] & H \end{tikzcd}$$

The lifting $\widetilde{F} : 0 \times \mathfrak{I} \to G$ in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r, "f"] \arrow[d, "i_0"'] & G \arrow[d, "p"] \\ 0 \times \mathfrak{I} \arrow[r, "F"'] \arrow[ur, "\widetilde{F}"'] & H \end{tikzcd}$$

is then determined by an arrow $g \in G$ satisfying $d(g) = x$ and $gp = h$. Therefore p is star surjective. $\square$

Now, we are assuming the same commutative square as in Proposition 3.1.1 with p as an ordered fibration and we want to define $\widetilde{F}$ to be the ordered covering homotopy. Recall that any morphism $E \times \mathfrak{I} \to G$ is completely determined by two morphisms $E \times 0 \to G$ and $\mathrm{Ob}(E) \times \mathfrak{I} \to G$ and for that, to construct $\widetilde{F}$ we need to define $\widetilde{F}$ first on $E \times 0$ and $\mathrm{Ob}(E) \times \mathfrak{I}$.

Define $\widetilde{F}$ on $E \times 0$ to be just f. Arrows of $\mathrm{Ob}(E) \times \mathfrak{I}$ are of the form $(e, \iota), (e, \iota^{-1})$ where

$e \in \mathrm{Ob}(E)$ and the ordering is determined by the poset $\mathrm{Ob}(E)$ as $(e, \iota) \leqslant (e', \iota) \iff e \leqslant e'$. $\widetilde{F}$ must satisfy $(e, \iota)\widetilde{F}p = (e, \iota)F$, $(e', \iota)\widetilde{F}p = (e', \iota)F$ and it must preserve the ordering in G, that is $(e, \iota)\widetilde{F} \leqslant (e', \iota)\widetilde{F}$ which is not ensured always. The following example shows how an ordered fibration does not imply an ordered covering homotopy.

Example 3.1.2. Let $E = e, f, z$ be a semilattice with order relations $e \geqslant z \leqslant f$. We let G to be a semilattice of groups over E, with ordering determined by two morphisms $\phi_e : G_e \to G_z$ and $\phi_f : G_f \to G_z$.
We also let H be a semilattice of groups over the semilattice $\{0, 1\}$, where $0 \leqslant 1$, with ordering determined by a morphism $\psi : H_1 \to H_0$.
Let $i : E \to G$ be the inclusion map, and let $p : G \to H$ be a fibration determined by three surjections $p_e : G_e \to H_1, p_f : G_f \to H_1, p_z : G_z \to H_0$ then

$$\begin{tikzcd}
G_e \arrow[rrd, "p_e"] \arrow[dd, "\phi_e"'] & G_f \arrow[rd, "p_f"] \arrow[ddl, "\phi_f"'] & \\
 & & H_1 \arrow[d, "\psi"] \\
G_z \arrow[rr, "p_z"'] & & H_0
\end{tikzcd}$$

is a commutative diagram.
The homotopy $F : E \times \mathfrak{I} \to H$ is then determined by $h_e, h_f \in H_1$ where $(e, \iota)F = h_e, (f, \iota)F = h_f, (z, \iota)F = h_e\psi = h_f\psi$.
A covering homotopy $\widetilde{F} : E \times \mathfrak{I} \to G$ is also determined by two elements $g_e \in G_e$ and $g_f \in G_f$ satisfying

- $g_e\phi_e = g_f\phi_f$,
- $g_ep_e = h_e$ and $g_fp_f = h_f$,
- $g_e\phi_ep_z = g_f\phi_fp_z = h_e\psi = h_f\psi$,

where
$(e, \iota)\widetilde{F} = g_e, (f, \iota)\widetilde{F} = g_f, (z, \iota)\widetilde{F} = g_e\phi_e = g_f\phi_f$.

Now, let $G_e = \mathbb{Z}$, $G_f = \mathbb{Z}_3 \times \mathbb{Z}_2$, $G_z = S_3$, $H_1 = \mathbb{Z}_6$, and $H_0 = \mathbb{Z}_2$ such that,

$\phi_e : \mathbb{Z} \to S_3$; where $(n)\phi_e = \begin{cases} (1) & \text{if } n \text{ is even,} \\ (12) & \text{if } n \text{ is odd} \end{cases}$
$\phi_f : \mathbb{Z}_3 \times \mathbb{Z}_2 \to S_3$; where
$(0, n) \to (1)$, $(1, n) \to (123)$, and $(2, n) \to (132)$

p_e is the epimorphism $\mathbb{Z} \to \mathbb{Z}_6$

p_f is the isomorphism $\mathbb{Z}_3 \times \mathbb{Z}_2 \to \mathbb{Z}_6$

and $p_z : S_3 \to \mathbb{Z}_2$, where $(\sigma)p_z = \begin{cases} 0 & \text{if } \sigma \text{ is even permutation} \\ 1 & \text{if } \sigma \text{ is odd permutation} \end{cases}$
Now, choose $h_e = 2$ and $h_f = 4$ from $\mathbb{Z}_6 = H_1$ then $2\psi = 4\psi = 0 = h_z$
this choice determines a homotopy $F : E \times \mathfrak{I} \to H$. Then $\widetilde{F} : E \times \mathfrak{I} \to G$ is determined by $g_e \in G_e = \mathbb{Z}$ and $g_f \in G_f = \mathbb{Z}_3 \times \mathbb{Z}_2$ where, $g_e p_e = 2$ and $g_f p_f = 4$. We find that

- $g_e \equiv 2 \pmod 6$ and $g_f = (1, 0)$,

- $g_e \phi_e = (1) \neq g_f \phi_f = (123)$,

- $g_e \phi_e p_z = 0 = g_f \phi_f p_z$

Definition An ordered morphism which has the covering homotopy property will be called a *strong (ordered) fibration*.

In the following proposition we prove a nice property of factorization for a strong fibration.

Proposition 3.1.3. *Suppose that a strong fibration $p : G \to H$ factors as the composite of a morphism $\pi : G \to T$ and an immersion $\psi : T \to H$. Then π is a strong fibration.*

Proof. Consider a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] & G \arrow[d, "\pi"] \\ A \times \mathfrak{I} \arrow[r, "F"'] & T \end{tikzcd}$$

Extend this diagram by ψ and use the homotopy lifting property of $p = \pi\psi$ to lift $F_* = F\psi$

to G

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] & G \arrow[d, "\pi"] \\ A \times \mathfrak{I} \arrow[ur, "\widetilde{F_*}"'] \arrow[r, "F"'] \arrow[dr, "F\psi"'] & T \arrow[d, "\psi"] \\ & H \end{tikzcd}$$

Then $i_0\widetilde{F_*} = f$. Consider $(a, \iota) \in A \times \mathfrak{I}$. Then
$d((a, \iota)\widetilde{F_*}\pi) = (d(a), 0)\widetilde{F_*}\pi = (d(a))f\pi = (d(a), 0)F = (d(a, \iota))F = d((a, \iota)F)$.
Now $(a, \iota)\widetilde{F_*}\pi\psi = (a, \iota)F\psi$, and since ψ is an immersion, $(a, \iota)\widetilde{F_*}\pi = (a, \iota)F$. Hence $\widetilde{F_*}$ lifts F. $\square$

3.2 Fibrations and Pullbacks in Ordered Groupoids

As we have seen in Proposition 2.1.3 that pullbacks exist in the category $\mathcal{OG}$, and in this section we show that the pullback of an ordered fibration is also an ordered fibration and the pullback of a strong fibration is a strong fibration.

Proposition 3.2.1. *(i) An ordered covering morphism $p : G \to H$ is a strong fibration.*

(ii) Given a pullback square of ordered groupoids

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "q"'] & G \arrow[d, "p"] \\ X \arrow[r, "f"'] & H \end{tikzcd}$$

if p is an ordered fibration (or strong fibration) then so also is q.

Proof. (i) We wish to construct $\widetilde{F}$ in a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] & G \arrow[d, "p"] \\ A \times \mathfrak{I} \arrow[ur, "\widetilde{F}"'] \arrow[r, "F"'] & H \end{tikzcd}$$

For $a \in A$ we must have $(a, 0)\widetilde{F} = af$. Now for any $x \in \mathrm{Ob}(A)$ we have $(x, 0)F = xi_0F = xfp$ and since p is a covering, there is a unique $g \in \mathrm{star}_G(xf)$ with $gp = (x, \iota)F$. We set $(x, \iota)\widetilde{F} = g$. Suppose that $y \leqslant x$ in $\mathrm{Ob}(A)$. Then $yf \leqslant xf$ and $(yf|g)p \leqslant gp = (x, \iota)F$ with $d(yf|g) = yf$. But also $(y, \iota)F \leqslant (x, \iota)F$ with $d((y, \iota)F) = yf$. By uniqueness of restriction we have $(y, \iota)\widetilde{F} = (yf|g) \leqslant g$. We have now defined $\widetilde{F}$ consistently on $A \times 0$, and on $\mathrm{Ob}(A) \times \mathfrak{I}$ these definitions combine to give $(a, \iota)\widetilde{F} = (af)g$ where g is the unique arrow in $\mathrm{star}_G(r(af))$ with $gp = (r(a), \iota)F$. This defines an ordered lifting $\widetilde{F}$ of F.

(ii) Suppose that p is a fibration and fix an identity (e, u) of Q then $ef = up$, and for any $x \in X$ starting at e we have xf starting at $ef = up$. Since p is a fibration, there is a $g \in G$ starting at u such that $gp = xf$. Hence $(x, g) \in Q$ and $(x, g)q = x$. Therefore q is a fibration. Now with p a strong fibration we want to show that q is a strong fibration. We can construct the map $\widetilde{F}$ to Q from maps to X and to G.

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "i_0"'] & Q \arrow[r, "\bar{f}"] \arrow[d, "q"] & G \arrow[d, "p"] \\
A \times \mathfrak{I} \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] \arrow[urr, dashed, "\phi"] & X \arrow[r, "f"'] & H
\end{tikzcd}
$$

we have $F : A \times \mathfrak{I} \to X$ and we lift $Ff : A \times \mathfrak{I} \to H$ to a map $\phi : A \times \mathfrak{I} \to G$. Since p is a strong fibration we have $\phi p = Ff$ and that leads to the following commutative diagram.

$$
\begin{tikzcd}
Q \arrow[drr, bend left, "\bar{f}"] \arrow[ddr, bend right, "q"'] & & \\
& A \times \mathfrak{I} \arrow[ul, dotted, "\widetilde{F}"'] \arrow[r, "\phi"'] \arrow[d, "F"] & G \arrow[d, "p"'] \\
& X \arrow[r, "f"] & H
\end{tikzcd}
$$

So $\widetilde{F}$ exists and q is a strong fibration.

$\square$

3.3 Fibrations and Exponential Law in Ordered Groupoids

In [3] Brown proves that the morphisms $Xp : \mathsf{GPD}(\mathsf{X}, \mathsf{G}) \to \mathsf{GPD}(\mathsf{X}, \mathsf{H})$ and $iH : \mathsf{GPD}(\mathsf{G}, \mathsf{H}) \to \mathsf{GPD}(\mathsf{A}, \mathsf{H})$ induced by the fibration $p : G \to H$ and the inclusion $i : A \to G$ respectively, are fibrations. In this section we study the same morphisms in the ordered case. We find that the ordered fibration p needs to satisfy the covering homotopy property to get Xp as an ordered fibration, and iH is not an ordered fibration in general. Supportive examples are provided.

Proposition 3.3.1. *Let $p : G \to H$ be a strong fibration. Then for any ordered groupoid X the induced ordered morphism*

$$Xp : \mathsf{OGPD}(X, G) \to \mathsf{OGPD}(X, H)$$

is a strong fibration.

Proof. The commutative square of ordered groupoids

$$\begin{tikzcd} E \arrow[r, "q"] \arrow[d, "i"'] & G \arrow[d, "p"] \\ E \times \mathfrak{I} \arrow[r, "F"'] & H \end{tikzcd}$$

has a covering homotopy $\widetilde{F}$ that makes the following square commute

$$\begin{tikzcd} E \arrow[r, "q"] \arrow[d, "i"'] & G \arrow[d, "p"] \\ E \times \mathfrak{I} \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & H \end{tikzcd}$$

Now, we have Xp defined on objects as $f(Xp) = fp$ such that

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "fp"] & \\ G \arrow[r, "p"'] & H \end{tikzcd}$$

Consider the commutative square

$$\begin{tikzcd} E \times X \arrow[r] \arrow[d] & G \arrow[d, "p"] \\ E \times \mathfrak{I} \times X \arrow[r] & H\,. \end{tikzcd}$$

Since p is assumed to satisfy the covering homotopy property, we have the covering homotopy $\widetilde{F} : E \times \mathfrak{I} \times X \to G$ which is an object in the ordered groupoid $\mathsf{OGPD}((E \times \mathfrak{I} \times X), G)$. And from the exponential law we find that $\widetilde{F}$ is an object in $\mathsf{OGPD}(E \times \mathfrak{I}, \mathsf{OGPD}(X, G))$, that is $\widetilde{F} : E \times \mathfrak{I} \to \mathsf{OGPD}(X, G)$ which is a covering homotopy in the square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathsf{OGPD}(X, G) \arrow[d, "Xp"] \\ E \times \mathfrak{I} \arrow[r] \arrow[ur, dashed, "\widetilde{F}"] & \mathsf{OGPD}(X, H) \end{tikzcd}$$

So, Xp is an ordered morphism that satisfies the covering homotopy property and for that it is a strong fibration. □

Remark 3.3.2. If we take p in the above proposition to be an ordered fibration without the covering homotopy property, then we can not ensure that the fibration Xp is ordered. The following example clarifies this matter of fact.

Example 3.3.3. Let $p : G \to H$ be as in Example 3.1.2 and take $X = X_1 \sqcup X_0$ with ordering morphism $\chi : X_1 \to X_0$. And let $\tau : \mathrm{Ob}(X) \to H$ be an ordered natural transformation such that: $X_1\tau \equiv 3 \pmod 6$ which implies $X_0\tau \equiv 1 \pmod 2$. Now, try to lift τ to an ordered natural transformation $\sigma : \mathrm{Ob}(X) \to G$ using the fibration p, then $X_1\sigma = (0,1)$ and $X_0\sigma = (12)$ but σ then would not be ordered since $X_0 \leqslant X_1$ but $(12) \nleqslant (0,1)$.

Given an ordered groupoid H and $i : A \to G$ an ordered inclusion of groupoids, the induced morphism $iH : \mathsf{OGPD}(G,H) \to \mathsf{OGPD}(A,H)$ is not an ordered fibration in general, the following example explains the case.

Example 3.3.4. Let G and H be the ordered groupoids in Example 3.1.2 and let $A = \mathbb{Z}_3 \times \mathbb{Z}_2 \sqcup S_3$. Define $\tau : \mathrm{Ob}(A) \to H$ to be the ordered natural transformation in which $(\mathbb{Z}_3 \times \mathbb{Z}_2)\tau \equiv 2 \pmod 6$ and $S_3\tau \equiv 0 \pmod 2$ If we extend $\tau : \mathrm{Ob}(G) \to H$ by taking $\mathbb{Z}\tau \equiv 1 \pmod 6$, then we have $S_3 \leqslant \mathbb{Z}$ but $S_3\tau \nleqslant \mathbb{Z}\tau$, So τ is not , in general, extendable to be ordered on G.

Remark 3.3.5. If A in the previous example is a wide subgroupoid of G, then the morphism iH is an ordered fibration.

Chapter 4

Actions and Semidirect Products of Ordered Groupoids

The category of actions of ordered groupoids $\mathcal{OACT}$ is described in the first section of this chapter with the considering of an ordered groupoid G acting on an ordered groupoid A. Then the ordered action groupoid $G \ltimes A$ has been constructed in the second section. The third section discusses the special case of ordered action when A is a poset. In addition, the functors $\sigma : \mathcal{OCOV} \to \mathcal{OACT}$ and $\omega : \mathcal{OACT} \to \mathcal{OCOV}$ are studied in the special case of A being a poset and we verify that σ is a right adjoint of ω. We also extend the factorization function in Corollary 2.3.17 to a functor $\mathcal{OFIB} \to \mathcal{OACT}$. Further, we link our work on quotients of ordered groupoids with the *maximum enlargement theorem*, we generalize this result by deriving it from an ordered functor that is not necessarily an immersion. Finally, we conclude this part by giving a direct and simplified construction of the derived ordered groupoid that follows the formulation in [22]. Then we prove Steinberg's *factorization theorem* [30] using our construction and link it to his construction of the derived ordered groupoid.

4.1 Ordered Groupoids Acting on Ordered Groupoids

The notions of a groupoid action on a groupoid and the associated construction of the semidirect product seem to have been first defined in [4]. We describe these notions for ordered groupoids and we develop some results of [4] for the ordered actions. In general, our definitions are equivalent to those given by Steinberg [30].

Definition An ordered groupoid G acts on an ordered groupoid A via an ordered functor $w : A \to \mathrm{Ob}(G)$, if for each $g \in G(x, y)$ and α in the ordered subgroupoid xw^{-1} there is given an element $\alpha \triangleleft g$ in the ordered subgroupoid yw^{-1}. This action must satisfy the following axioms:

1. If gh defined in G and $\alpha \triangleleft g$ exists, then $(\alpha \triangleleft g) \triangleleft h = \alpha \triangleleft gh$;
2. If α and β are composable elements of the ordered groupoid xw^{-1} and $\alpha \triangleleft g$ exists, then $(\beta\alpha) \triangleleft g = (\beta \triangleleft g)(\alpha \triangleleft g)$;
3. If $e \in \mathrm{Ob}(G)$ with $\alpha \in ew^{-1}$, then $\alpha \triangleleft e = \alpha$;
4. If $\alpha \triangleleft g$ and $\beta \triangleleft h$ are defined, and $\alpha \leqslant \beta$ in A, $g \leqslant h$ in G then $\alpha \triangleleft g \leqslant \beta \triangleleft h$.

In such a situation, we call A a *G-groupoid*.

Remark 4.1.1. Since $\mathrm{Ob}(G)$ is a poset and w is an ordered morphism, the ordered groupoid A is the sum $A = \bigsqcup_x A_x$ of the ordered subgroupoids $A_x = xw^{-1}$ for $x \in \mathrm{Ob}(G)$.

An element $g \in G(x, y)$ defines a morphism of groupoids $g_* : A_x \to A_y$ given by $\alpha \mapsto \alpha \triangleleft g$ such that:

- If $e \in \mathrm{Ob}(G)$, then $e_* = 1 : A_e \to A_e$ the identity ;
- If gh is defined in G, then $(gh)_* = g_* h_*$.

and since the action preserves the ordering, the morphism g_* is ordered. Thus, an action of G on A defines a functor $A' : G \to \mathcal{OG}$ from the ordered groupoid G to the category of ordered groupoids such that $xA' = A_x$ for $x \in \mathrm{Ob}(G)$ and $gA' = g_*$ for g an element in G. On the other hand, if we have a functor $A' : G \to \mathcal{OG}$ then we can obtain an action of G on the sum of ordered groupoids xA' for $x \in \mathrm{Ob}(G)$ where there is no ordering between xA' and yA' for $x \neq y$.

The Category of Actions of Ordered Groupoids $\mathcal{OACT}$

The actions of ordered groupoids form a category $\mathcal{OACT}$ whose objects are actions of ordered groupoids $(G, A, w, \triangleleft)$ and whose arrows $(G, A, w, \triangleleft) \to (G', A', w', \triangleleft)$ are pairs (ψ, f) where $\psi : G \to G'$ and $f : A \to A'$ are ordered morphism of groupoids, such that:

- $fw' = w\,\mathrm{Ob}(\psi)$
- $\alpha f \triangleleft g\psi = (\alpha \triangleleft g)f$, whenever $\alpha \triangleleft g$ is defined.

The composition of arrows is defined as the usual composition componentwise on each ordered functor of the pair (ψ, f) and for each $(G, A, w, \triangleleft)$ in $\mathrm{Ob}(\mathcal{OACT})$, there is an identity arrow consisting of $1_G : G \to G$ and $1_A : A \to A$.

4.2 Split Extensions

In this section we describe the construction of the split extension (or the semidirect product), $G \ltimes A$ for the case that G is an ordered groupoid acting on the ordered groupoid A via $w : A \to \mathrm{Ob}(G)$. We first set

$$\mathrm{Ob}(G \ltimes A) = \mathrm{Ob}(A)\,.$$

Next if $a, b \in \mathrm{Ob}(A)$, we define $(G \ltimes A)(a, b) = \{(g, \alpha) : g \in G(aw, bw), \alpha \in A(a \triangleleft g, b)\}$

$$\begin{tikzcd}
 & b \\
a & \\
 & a \triangleleft g \arrow[uu, "\alpha"] \\
aw \arrow[r, "g"'] & bw
\end{tikzcd}$$

The composition of $(g, \alpha) \in (G \ltimes A)(a, b)$ and $(h, \beta) \in (G \ltimes A)(b, c)$ is defined as follows:

$$(g, \alpha)(h, \beta) = (gh, (\alpha \triangleleft h)\beta)$$

$$\begin{tikzcd}
 & & c \\
 & b & b \triangleleft h \arrow[u, "\beta"'] \\
a & a \triangleleft g \arrow[u, "\alpha"] & a \triangleleft gh \arrow[u, "\alpha \triangleleft h"'] \\
aw \arrow[r, "g"'] & bw \arrow[r, "h"'] & cw
\end{tikzcd}$$

(g, α) has right identity of the form $(1_{bw}, 1_b)$ and left identity of the form $(1_{aw}, 1_a)$. The inverse of (g, α) is $(g^{-1}, \alpha^{-1} \triangleleft g^{-1})$

$$\begin{tikzcd}
b \triangleleft g^{-1} \arrow[d, "\alpha^{-1} \triangleleft g^{-1}"] & b \\
a & \\
aw & bw \arrow[l, "g^{-1}"]
\end{tikzcd}$$

Since the associativity is easily verified, we have $G \ltimes A$ as a groupoid and the ordering of this groupoid is defined as follows:

If $g \in G(aw, bw)$, $\alpha \in A(a \triangleleft g, b)$ and $h \in G(cw, dw)$, $\beta \in A(c \triangleleft h, d)$ then we say that

$$(g, \alpha) \leqslant (h, \beta) \iff g \leqslant h \text{ in } G, \alpha \leqslant \beta \text{ in } A \text{ and } a \leqslant c \text{ in } \mathrm{Ob}(A).$$

We now verify the ordered groupoid axioms for the groupoid $G \ltimes A$:

OG1 If $(g, \alpha) \leqslant (h, \beta)$ then $g \leqslant h, \alpha \leqslant \beta$ and $a \leqslant c$. Since G and A are ordered groupoids, we have $g^{-1} \leqslant h^{-1}, \alpha^{-1} \leqslant \beta^{-1}$ and that implies $\alpha^{-1} \triangleleft g^{-1} \leqslant \beta^{-1} \triangleleft h^{-1}$, we also have $b \leqslant d$ implied by $\alpha \leqslant \beta$. So $(g^{-1}, \alpha^{-1} \triangleleft g^{-1}) \leqslant (h^{-1}, \beta^{-1} \triangleleft h^{-1})$.

OG2 If $(g, \alpha) \leqslant (h, \beta)$ and $(f, \gamma) \leqslant (k, \rho)$ such that $(g, \alpha) \in (G \ltimes A)(a, b)$, $(h, \beta) \in (G \ltimes A)(c, d)$, $(f, \gamma) \in (G \ltimes A)(b, e)$ and $(k, \rho) \in (G \ltimes A)(d, n)$. Then by the ordering of G and A we have $gf \leqslant hk$, $(\alpha \triangleleft f) \leqslant (\beta \triangleleft k)$ and $(\alpha \triangleleft f)\gamma \leqslant (\beta \triangleleft k)\rho$. And so $(gf, (\alpha \triangleleft f)\gamma) \leqslant (hk, (\beta \triangleleft k)\rho)$.

OG3 If $(g, \alpha) \in (G \ltimes A)(a, b)$ and $e \leqslant a$ in $\mathrm{Ob}(A)$, then $ew \leqslant aw$ in G and there is a unique element $(ew|g) \leqslant g$ in G. We also have $e \triangleleft g \leqslant a \triangleleft g$ so there is a unique element $(e \triangleleft g|\alpha) \leqslant \alpha$ in A. That means we have a unique element $(ew|g, e \triangleleft g|\alpha) \leqslant (g, \alpha)$ in $G \ltimes A$ starts at e and it is the restriction $(e|(g, \alpha))$.

The ordered projection $p : G \ltimes A \to G$ is defined to be w on objects, and on elements as $(g, \alpha) \mapsto g$. The ordered injection $i : A \to G \ltimes A$ is defined on objects to be the identity and on elements as $\alpha \mapsto (\alpha w, \alpha)$. Clearly, $Im(i) = \ker p$.

Proposition 4.2.1. *p is an ordered strong fibration of groupoids.*

Proof. Given a commutative square

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i_0"'] & G \ltimes A \arrow[d, "p"] \\ B \times \mathfrak{I} \arrow[r, "F"'] & G \end{tikzcd}$$

in which $bf = (g_b, a_b) \in G \ltimes A$ and $(y, \iota)F = h_y \in G$, we define

$$(b, \iota)\widetilde{F} = (g_b h_{r(b)}, a_b \triangleleft h_{r(b)})$$

$\widetilde{F}$ is an ordered functor lifting F. □

Remark 4.2.2. When A is discrete, p is an ordered covering morphism.

Now we discuss the ordering on $\pi_0 A$ in order to obtain an action of G on $\pi_0 A$ from the action of G on A. As we have seen in Example 2.3.18 $\pi_0 A$ is a preordered set and the canonical quotient $Q(A)$ is the poset isomorphic to $A/_{\leqslant} A$. We denote $Q(A)$ here by $\overline{\pi}_0 A$.

The action of G on A determines two ordered actions of G on posets; the action of G on $\mathrm{Ob}(A)$ which is the ordered action of ordered groupoid on a poset, discussed later in this chapter, and the action of G on $\overline{\pi}_0 A$, which is tied with the ordered projection $p : G \ltimes A \to G$, to explain this recall that p has the factorization:

$$
\begin{tikzcd}
G \ltimes A \arrow[dr, "\varpi"] \arrow[dd, "p"'] & \\
& G \ltimes A/_{\leqslant} \ker p \arrow[dl, "\rho"] \\
G &
\end{tikzcd}
$$

and ρ is an ordered covering, then we have the following result.

Proposition 4.2.3. *There is an ordered isomorphism*

$$(G \ltimes A)/_{\leqslant} \ker p \to G \ltimes \overline{\pi}_0 A$$

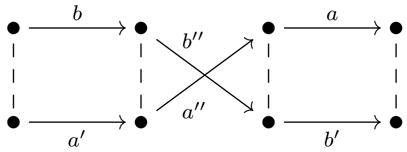
Proof. Before we start the proof it is helpful to recall and identify the notation used in this proof:
$\bar{a}$ denotes the connected component of $a \in A$ in $\pi_0 A$ and $[\bar{a}]$ denotes the equivalence class in $\overline{\pi}_0 A$ constructed from the preorder on $\pi_0 A$

- G acts on $\overline{\pi}_0 A$:
  Since $w : A \to \mathrm{Ob}(G)$ is a functor to a discrete groupoid $\mathrm{Ob}(G)$, it must be constant on connected components, that is if $a \in \bar{b}$ then $aw = bw$. Hence w induces an ordered mapping $w_0 : \pi_0 A \to \mathrm{Ob}(G)$ given by $\bar{a} \mapsto aw$ and since $\mathrm{Ob}(G)$ is a poset, this w_0 in turn induces $\bar{w} : \overline{\pi}_0 A \to \mathrm{Ob}(G)$ mapping $[\bar{a}] \mapsto aw$.
  The action of $g \in G$ on $[\bar{a}] \in \overline{\pi}_0 A$ is defined as $[\bar{a}] \triangleleft g = [\overline{a \triangleleft g}]$.
  If $[\bar{a}] \leqslant [\bar{b}]$ then for each $b' \in [\bar{b}]$ there exists $a' \in [\bar{a}]$ with $a' \leqslant b'$. If $g \leqslant h$ in G then $a' \triangleleft g \leqslant b' \triangleleft h$ and so $[\overline{a \triangleleft g}] \leqslant [\overline{b \triangleleft h}]$.

- The semidirect product $G \ltimes \overline{\pi}_0 A$ exists and there is a groupoid morphism $q : G \ltimes A \to G \ltimes \overline{\pi}_0 A$ mapping $(g, a) \mapsto (g, [\bar{a}])$.
  Recall that $\ker p = \{(g, a) \in G \ltimes A : g \in \mathrm{Ob}(G)\} = \{(aw, a) : a \in A\}$ and suppose that $(g, a) \simeq_{\ker p} (h, b)$, so $(g, a) \simeq_{\ker p} (h, b) \iff$ there exist $a_1, a_2, b_1, b_2 \in A$ such that $(a_1 w, a_1)(g, a)(a_2 w, a_2) \leqslant (h, b)$ and $(b_1 w, b_1)(h, b)(b_2 w, b_2) \leqslant (g, a)$
  But $(a_1 w, a_1)(g, a)(a_2 w, a_2) = (g, (a_1 \triangleleft g) a a_2)$, hence $g \leqslant h$ and $(a_1 \triangleleft g) a a_2 \leqslant b$.
  Similarly, $h \leqslant g$ and $(b_1 \triangleleft h) b b_2 \leqslant a$
  Now $(g, a)q = (g, [\bar{a}]), (h, b)q = (h, [\bar{b}])$ and $h = g$ since $g \leqslant h, h \leqslant g$. We also have $((a_1 \triangleleft g) a a_2) \in \bar{a}$ and $((a_1 \triangleleft g) a a_2) \leqslant b$ so $\bar{a} \leqslant \bar{b}$ in $\pi_0 A$, similarly $\bar{b} \leqslant \bar{a}$ and we have $[\bar{a}] = [\bar{b}]$ which means $(g, a)q = (h, b)q$. Hence q induces

$$\bar{q} : G \ltimes A/_{\leqslant} \ker p \to G \ltimes \overline{\pi}_0 A$$

- To prove that $\bar{q}$ is an isomorphism, suppose that $(g, a)q = (h, b)q$ then $g = h$ and $[\bar{a}] = [\bar{b}]$. Hence $\bar{a} \leqslant \bar{b}$ and $\bar{b} \leqslant \bar{a}$ and so there exist $a' \in \bar{a}$ with $a' \leqslant b$ and $b' \in \bar{b}$ with $b' \leqslant a$.



$a' = a'a''aa^{-1}a''^{-1} = ((a'a'') \triangleleft g^{-1} \triangleleft g)a(a^{-1}a''^{-1}) \leqslant b$ since $aw = r(g)$ so g^{-1} acts on the component $\bar{a}$. Similarly, $b' = b''^{-1}b^{-1}bb''b' = ((b''^{-1}b^{-1}) \triangleleft h^{-1} \triangleleft h)b(b''b') \leqslant a$ and from the characterization of $\simeq_{\ker p}$ above we see that $(g, a) \simeq_{\ker p} (h, b)$. Hence $(g, a)q = (h, b)q \iff (g, a) \simeq_{\ker p} (h, b)$ and $\bar{q} : G \ltimes A/_{\leqslant} \ker p \to G \ltimes \overline{\pi}_0 A$ is an isomorphism of ordered groupoids.

$\square$

The following result is an ordered version of [4, Proposition 2.6] (which Brown attributes to A. Fröhlich).

Proposition 4.2.4. *The ordered morphisms*

$$A \xrightarrow{\quad i \quad} G \ltimes A \xrightarrow{\quad p \quad} G$$

have the following properties:

(i) (i) *is an isomorphism onto* $\ker p$.

(ii) $G \ltimes A$ *contains* $G \ltimes \mathrm{Ob}(A)$ *as a wide subgroupoid, and* $p|G \ltimes \mathrm{Ob}(A)$ *is the covering projection and* $\mathrm{Ob}(i)$ *is the identity.*

(iii) if $\alpha \in A(a \triangleleft g, b)$, *where* $g \in G(aw, bw)$ *then we have*

$$(g, a \triangleleft g)(\alpha i)(g^{-1}, b \triangleleft g^{-1}) = (\alpha \triangleleft g^{-1})i.$$

(iv) The above three properties characterize the triple $(G \ltimes A, i, p)$ *up to isomorphism.*

(v) Each element $\omega \in G \ltimes A$ *has a unique representation* $\omega = \mu(\alpha i)$ *with* $\alpha \in A$ *and* $\mu \in G \ltimes \mathrm{Ob}(A)$.

Proof. (i) As we have seen $\ker p = \{(\alpha w, \alpha) : \alpha \in A\}$, each element $(\alpha w, \alpha)$ in $\ker p$ is just an image of a unique element $\alpha \in A$ by i.

(ii) p is proved to be a strong fibration in general in Proposition 4.2.1, so let $(g, a \triangleleft g) \in (G \ltimes \mathrm{Ob}(A))(a, b)$ and $(h, a \triangleleft h) \in (G \ltimes \mathrm{Ob}(A))(a, c)$ where $a \triangleleft g = b$ and $a \triangleleft h = c$ and suppose that $(g, a \triangleleft g)p = (h, a \triangleleft h)p$ that is $g = h$ and that implies $a \triangleleft g = a \triangleleft h$, so $p|G \ltimes \mathrm{Ob}(A)$ is star injective.

(iii) $(g, a \triangleleft g)(\alpha i)(g^{-1}, b \triangleleft g^{-1}) = (g, a \triangleleft g)(\alpha w, \alpha)(g^{-1}, b \triangleleft g^{-1}) = (g, \alpha)(g^{-1}, b \triangleleft g^{-1}) = (aw, (\alpha b) \triangleleft g^{-1}) = ((\alpha \triangleleft g^{-1})w, \alpha \triangleleft g^{-1}) = (\alpha \triangleleft g^{-1})i$.

(iv) Suppose given ordered morphisms:

$$A \xrightarrow{i'} E \xrightarrow{p'} G$$

satisfying the above three properties, we wish to define an ordered isomorphism $\theta : G \ltimes A \to E$ such that $i\theta = i'$ and $\theta p' = p$.

$$\begin{tikzcd}
 & G \ltimes A \arrow[dr, "p"] \arrow[dd, "\theta"] & \\
A \arrow[ur, "i"] \arrow[dr, "i'"'] & & G \\
 & E \arrow[ur, "p'"'] &
\end{tikzcd}$$

Since $\mathrm{Ob}(E) = \mathrm{Ob}(A)$, let θ be the identity on objects and define $(g, \alpha)\theta = (g, a \triangleleft g)(\alpha i')$ where $(g, \alpha) \in (G \ltimes A)(a, b)$.

To prove the injectivity of θ, we know that $\mathrm{Ob}(\theta) = 1$ so if $(g, \alpha) \in (G \ltimes A)(a, b)$ and $(g', \alpha') \in (G \ltimes A)(a', b')$ with $(g, \alpha)\theta = (g', \alpha')\theta$, then we should have $a = a'$ and $b = b'$. Now, if $[(g^{-1}, \alpha^{-1} \triangleleft g^{-1})(g', \alpha')]\theta$ is an identity then $(g^{-1}g', (\alpha^{-1} \triangleleft g^{-1}g')\alpha')\theta p'$ is also an identity and that is $g^{-1}g'$ is an identity so $g = g'$. But $(g^{-1}g, \alpha^{-1}\alpha')\theta = (bw, b)(\alpha^{-1}\alpha')i'$ and that implies $(\alpha^{-1}\alpha')i'$ is an identity, so as $\alpha^{-1}\alpha'$ and that is $\alpha = \alpha'$.

For surjectivity, let $\zeta \in E(a, b)$ with $\zeta p' = g$ then $((g^{-1}, a)\theta\zeta)p' = (g^{-1}, a)\theta p'\zeta p' = g^{-1}g = bw$, that means $((g^{-1}, a)\theta\zeta) \in \ker p'$ so we can say $(g^{-1}, a)\theta\zeta = \alpha i'$ for some $\alpha \in A(a \triangleleft g, b)$ and we have $\zeta = ((g^{-1}, a)\theta)^{-1}(\alpha i') = (g^{-1}, a)^{-1}\theta(\alpha i\theta) = ((g, a \triangleleft g)(\alpha i))\theta = (g, \alpha)\theta$.

θ is a morphism for

$$\begin{aligned}(gh, \alpha \triangleleft h\beta)\theta &= (gh, a \triangleleft gh)(\alpha \triangleleft h\beta)i' \\ &= (gh, c)(\alpha \triangleleft h)i'(\beta)i' \\ &= (\alpha \triangleleft hh^{-1}g^{-1})i'(gh, c)(\beta)i' \\ &= (\alpha \triangleleft g^{-1})i'(gh, c)(\beta)i' \\ &= (\alpha \triangleleft g^{-1})i'(g, b)(h, c)(\beta)i' \\ &= (g, a \triangleleft g)(\alpha)i'(h, b \triangleleft h)(\beta)i' \\ &= (g, \alpha)\theta(h, \beta)\theta\end{aligned}$$

where $(g, \alpha) \in (G \ltimes A)(a, b)$ and $(h, \beta) \in (G \ltimes A)(b, c)$.
Last thing is to show that θ is ordered, for that let $(g, \alpha) \leqslant (h, \beta)$ such that $(g, \alpha) \in (G \ltimes A)(a, b)$ and $(h, \beta) \in (G \ltimes A)(c, d)$ then $g \leqslant h \in G$, $\alpha \leqslant \beta \in A$ and $a \leqslant c, b \leqslant d$. We now have $(g, \alpha)\theta = (g, a \triangleleft g)(\alpha i') \leqslant (h, \beta)\theta = (h, c \triangleleft h)(\beta i')$ from the assumption that i' is ordered.

(v) $\omega = (g, \alpha) = (g, a \triangleleft g)(bw, \alpha) = (g, a \triangleleft g)(\alpha)i$ where $(g, a \triangleleft g) \in G \ltimes \mathrm{Ob}(A)$ and $\alpha \in A$.

□

Definition Let A be a G-groupoid via the ordered morphism $w : A \to \mathrm{Ob}(G)$ and A' a G'-groupoid via $w' : A' \to \mathrm{Ob}(G')$. A morphism of split extensions $G \ltimes A \to G' \ltimes A'$ is a pair (ψ, f) such that $\psi : G \to G'$ and $f : A \to A'$ are ordered morphisms of groupoids with the following axioms:

1. $fw' = w\psi$.
2. $(\alpha \triangleleft g)f = (\alpha f) \triangleleft g\psi$.

This morphism (ψ, f) induces the commutative square

$$\begin{tikzcd} G \ltimes A \arrow[r, "\psi \ltimes f"] \arrow[d, "p"'] & G' \ltimes A' \arrow[d, "p'"] \\ G \arrow[r, "\psi"'] & G' \end{tikzcd}$$

It is worthwhile to expose the functor $F : \mathcal{OACT} \to \mathcal{OG}$ from the category of ordered actions of groupoids to the category of ordered groupoids such that F is defined on objects as

$$(G, A, w, \triangleleft)F = G \ltimes A,$$

and on arrows as

$$(\psi, f)F = \psi \ltimes f$$

where $(\psi, f) : (G, A, w, \triangleleft) \to (G', A', w', \triangleleft)$ and

$$\psi \ltimes f : G \ltimes A \to G' \ltimes A'.$$

As a special case, let $G = G'$ and $\psi = 1_G$, then $f : A \to A'$ is an ordered morphism of G-groupoids. Certain properties of f are preserved under $f \mapsto (1, f)$. The following result is an ordered version of [4, Proposition 2.7], and is closely related to [30, Theorem 4.7].

Proposition 4.2.5. *If f satisfies any of the following properties then so does $1 \ltimes f$, namely:*

(i) injective,

(ii) fibration,

(iii) connected fibres,

(iv) quotient mapping,

(v) discrete kernel,

(vi) covering morphism.

Proof. (i) $(1 \ltimes f) : G \ltimes A \to G \ltimes A'$, let $(g, \alpha)(1 \ltimes f) = (h, \beta)(1 \ltimes f)$ meaning $(g, \alpha f) = (h, \beta f)$ that is $g = h \in G$ and $\alpha f = \beta f \in A'$, but we assumed f to be injective so $\alpha = \beta \in A$.

(ii) Let $a \in \mathrm{Ob}(A)$ and let $(g, \beta) \in G \ltimes A'(af, b)$, then $\beta \in A'(af \triangleleft g, b) = A'((a \triangleleft g)f, b)$. Since f is a fibration, there is an $\alpha \in A(a \triangleleft g, c)$ such that $\alpha f = \beta$ then $(g, \alpha) \in (G \ltimes A)(a, c)$ is a lift of (g, β) starting at a. Thus $1 \ltimes f$ is a fibration.

(iii) If f has connected fibres then for $a, b \in \mathrm{Ob}(A)$ satisfying $af = bf$ we have an $\alpha \in A(a, b)$ with $\alpha f = af = bf$, whence $aw = bw$ and we have $(aw, \alpha) \in (G \ltimes A)(a, b)$.

(iv) If (ii) and (iii) are satisfied together, that is equivalent to (iv).

(v) Let $a \in \mathrm{Ob}(A)$ and $(g, \alpha)(1 \ltimes f) = (h, \beta)(1 \ltimes f)$ in $(G \ltimes A')(af, -)$, then we have $(g, \alpha f) = (h, \beta f) \in G \ltimes A'(af, -)$ and that is $g = h \in G$ and $\alpha f = \beta f \in A'((a \triangleleft g)f, -)$, but we know that f has discrete kernel which means $\alpha = \beta \in A(a \triangleleft g, -)$ and $(g, \alpha) = (h, \beta)$ in $(G \ltimes A)(a, -)$.

(vi) If (ii) and (v) are satisfied together, that is equivalent to (vi).

□

4.3 Ordered Groupoids Acting on Posets

In this section we deal with an importent special case of actions of ordered groupoids, that is when A is a discrete ordered groupoid (a poset). Let G be an ordered groupoid. A right ordered action of G on the poset A is a quadruple $(G, A, w, \triangleleft)$ where $w : A \to \mathrm{Ob}(G)$ is an ordered function and $\triangleleft : (a, g) \to a \triangleleft g$ is an ordered function defined whenever $g \in \mathrm{star}_G(aw)$ with the axioms

- $(a \triangleleft g)w = r(g)$,
- $a \triangleleft 1 = a$,
- $(a \triangleleft g) \triangleleft g' = a \triangleleft (gg')$ whenever both sides are defined.

If $g \in G(x, y)$, then the action gives an ordered function $g_* : (x)w^{-1} \to (y)w^{-1}, a \mapsto a \triangleleft g$, and we have $1_* = 1$, $(gg')_* = g_* g'_*$. Thus an action of G on the poset A defines a functor from the ordered groupoid G to the category of posets $G \to \mathcal{POSET}$.
The action groupoid, which is the semidirect product is also a special case of $G \ltimes A$ when both G and A are ordered groupoids. It is described as follows:
$\mathrm{Ob}(G \ltimes A) = A$ and $G \ltimes A(a, b) = \{(g, b) : g \in G(aw, bw), b = a \triangleleft g\}$. The composition is defined as $(g, b)(h, c) = (gh, c)$ such that $b \triangleleft h = c$ and $(g, b) \in (G \ltimes A)(a, b), (h, c) \in (G \ltimes A)(b, c)$. The inverse $(g, b)^{-1}$ is of the form $(g^{-1}, b \triangleleft g^{-1})$. The ordering is inherited from the general case, componentwise.
The ordered projection $p : G \ltimes A \to G$; $(g, b) \to g$ is an ordered covering as we have seen in Proposition 4.2.4 when A is a poset.

The Functors $\mathcal{OCOV} \to \mathcal{OACT}$ and $\mathcal{OACT} \to \mathcal{OCOV}$

A functor $\mathcal{COV} \to \mathcal{ACT}$ from the category of covering morphisms of groupoids to the category of actions of groupoids has been constructed in [3], and using the same argument we construct a functor $\sigma : \mathcal{OCOV} \to \mathcal{OACT}$ from the category of ordered coverings of groupoids to the category of ordered actions of groupoids.

Let $q : G \to H$ be an ordered covering morphism of ordered groupoids. We can define an ordered action of H on $\mathrm{Ob}(G)$ as follows. Let $h \in H(x, y)$ and $x' \in xq^{-1}$, the action of h on x' is determined by the unique element g of $\mathrm{star}_G(x')$ that lifts h where $r(g) = x' \triangleleft h$. So we have for each ordered covering morphism $q : G \to H$ an ordered action $(H, \mathrm{Ob}(G), \mathrm{Ob}(q), \triangleleft)$. Now, the function $q \to (H, \mathrm{Ob}(G), \mathrm{Ob}(q), \triangleleft)$ can be extended to a functor $\sigma : \mathcal{OCOV} \to \mathcal{OACT}$ where σ is defined on objects as

$$q\sigma = (H, \mathrm{Ob}(G), \mathrm{Ob}(q), \triangleleft),$$

and to define it on arrows consider the following commutative square

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "q"'] & G' \arrow[d, "q'"] \\ H \arrow[r, "\psi"'] & H' \end{tikzcd}$$

q, q' are objects of $\mathcal{OCOV}$ and $(\phi, \psi) \in \mathcal{OCOV}(q, q')$ where ϕ and ψ are ordered morphisms of groupoids. If we take f to be $\mathrm{Ob}(\phi)$ then we have:

$$f\,\mathrm{Ob}(q') = \mathrm{Ob}(q)\,\mathrm{Ob}(\psi)$$

which is the first condition for (ψ, f) to be an arrow in $\mathcal{OACT}$. Now, let $h \in H(x, y)$ and $x' \in xq^{-1}$. if $g \in G(x', y')$ lifts h, then $x' \lhd h = y'$ and $g\phi : x'f \to y'f$ lifts $h\psi$. so we have:

$$x'f \lhd h\psi = y'f = (x' \lhd h)f$$

which is the second condition for (ψ, f) to be an arrow in $\mathcal{OACT}$. The verification of σ to be a functor is straightforward from the following commutative diagrams:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "q"'] & G' \arrow[r, "\phi'"] \arrow[d, "q'"'] & \bar{G} \arrow[d, "\bar{q}"'] \\ H \arrow[r, "\psi"'] & H' \arrow[r, "\psi'"'] & \bar{H} \end{tikzcd} \quad \overset{\sigma}{\curvearrowright} \quad \begin{tikzcd} \mathrm{Ob}(G) \arrow[r, "f"] \arrow[d, "\mathrm{Ob}(q)"'] & \mathrm{Ob}(G') \arrow[r, "f'"] \arrow[d, "\mathrm{Ob}(q')"'] & \mathrm{Ob}(\bar{G}) \arrow[d, "\mathrm{Ob}(\bar{q})"'] \\ H \arrow[r, "\psi"'] & H' \arrow[r, "\psi'"'] & \bar{H} \end{tikzcd}$$

and

$$\begin{tikzcd} G \arrow[r, "1_G"] \arrow[d, "q"'] & G \arrow[d, "q"] \\ H \arrow[r, "1_H"'] & H \end{tikzcd} \quad \overset{\sigma}{\curvearrowright} \quad \begin{tikzcd} \mathrm{Ob}(G) \arrow[r, "1_{\mathrm{Ob}(G)}"] \arrow[d, "\mathrm{Ob}(q)"'] & \mathrm{Ob}(G) \arrow[d, "\mathrm{Ob}(q)"] \\ H \arrow[r, "1_H"'] & H \end{tikzcd}$$

We now aim to construct a functor $\omega : \mathcal{OACT} \to \mathcal{OCOV}$ starting with an ordered action of G on the poset A, $(G, A, w, \lhd)$ and applying F to this action gives the ordered groupoid $G \ltimes A$. As we have seen, the mapping $p : G \ltimes A \to G$ is an ordered covering morphism of groupoids since A is discrete, so we can define a function $(G, A, w, \lhd) \to (p : G \ltimes A \to G)$ and extend this function to the functor ω by using the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] & A' \arrow[d, "i'_0"] \\ G \ltimes A \arrow[r, "\phi"] \arrow[d, "p"'] & G' \ltimes A' \arrow[d, "p'"] \\ G \arrow[r, "\psi"] & G' \end{tikzcd}$$

where, (f, ψ) is an arrow in $\mathcal{OACT}$; $\phi = \psi \ltimes f$ is an ordered morphism of groupoids , i_0, i_0' are ordered inclusions, and p, p' are the ordered covering projections defined earlier. ω is defined on objects of $\mathcal{OACT}$ as $(G, A, w, \triangleleft)\omega = p$ and on arrows of $\mathcal{OACT}$ as $(f, \psi)\omega = (\phi, \psi)$

It is clear and straightforward from the following diagram that

$$(ff', \psi\psi')\omega = (f, \psi)\omega(f', \psi')\omega = (\phi, \psi)(\phi', \psi').$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_0"'] & A' \arrow[r, "f'"] \arrow[d, "i_0'"] & \bar{A} \arrow[d, "\bar{i}"] \\
G \ltimes A \arrow[r, "\phi"] \arrow[d, "p"'] & G' \ltimes A' \arrow[r, "\phi'"] \arrow[d, "p'"] & \bar{G} \ltimes \bar{A} \arrow[d, "\bar{p}"] \\
G \arrow[r, "\psi"] & G' \arrow[r, "\psi'"] & \bar{G}
\end{tikzcd}$$

consequently, $(\mathrm{id}_A, \mathrm{id}_G)\omega = (\mathrm{id}_G \ltimes \mathrm{id}_A, \mathrm{id}_G)$. We now have the composition $\mathcal{OCOV} \longrightarrow \mathcal{OACT} \longrightarrow \mathcal{OCOV}$; mapping

$$q \mapsto (H, \mathrm{Ob}(G), \mathrm{Ob}(q), \triangleleft) \mapsto p$$

such that $q : G \to H$ is an ordered covering by assumption and $p : H \ltimes \mathrm{Ob}(G) \to H$ is an ordered covering by Proposition 4.2.4. For each ordered covering morphism $p : G \ltimes A \to G$ there is an ordered action $(G, (G \ltimes A)_0, \mathrm{Ob}(p), \triangleleft)$ which determines an ordered action groupoid $G \ltimes (G \ltimes A)_0 = G \ltimes A$ so, we also have the composition $\mathcal{OACT} \longrightarrow \mathcal{OCOV} \longrightarrow \mathcal{OACT}$ such that

$$(G, A, w, \triangleleft) \mapsto p \mapsto (G, (G \ltimes A)_0, \mathrm{Ob}(p), \triangleleft)$$

and that leads to the following proposition.

Proposition 4.3.1. *$\sigma : \mathcal{OCOV} \to \mathcal{OACT}$ is a right adjoint of $\omega : \mathcal{OACT} \to \mathcal{OCOV}$.*

Proof. It suffices to prove that the mapping

$$\mathcal{OACT}((K, X, w, \triangleleft), (H, \mathrm{Ob}(G), \mathrm{Ob}(q), \triangleleft)) \longrightarrow \mathcal{OCOV}(p, q)$$

is a bijection. Let $(\phi, \psi) \in \mathcal{OCOV}(p, q)$ which makes the following diagram commute

$$\begin{tikzcd}
K \ltimes X \arrow[r, "\phi"] \arrow[d, "p"'] & G \arrow[d, "q"] \\
K \arrow[r, "\psi"'] & H
\end{tikzcd}$$

if we take f to be $\mathrm{Ob}(\phi)$ then we have $f\,\mathrm{Ob}(q) = \mathrm{Ob}(p)\,\mathrm{Ob}(\psi)$ which is the first condition for (ψ, f) to be in $\mathcal{OACT}((K, X, w, \triangleleft), (H, \mathrm{Ob}(G), \mathrm{Ob}(q), \triangleleft))$. Take $k \in K(x, y)$ with

$x' \in xq^{-1}$ and let (k, x') the unique element in $(K \ltimes X)(x', y')$ that lifts k, then $x' \triangleleft k = y'$ and $(k, x')\phi$ is an arrow from $x'f$ to $y'f$ in G which lifts $k\psi$, so we have $x'f \triangleleft k\psi = y'f = (x' \triangleleft k)f$ and that is the second condition for (ψ, f) to be in

$$\mathcal{OACT}((K, X, w, \triangleleft), (H, \mathrm{Ob}(G), \mathrm{Ob}(q), \triangleleft))$$

that means for each $(\phi, \psi) \in \mathcal{OCOV}(p, q)$ there is

$$(\psi, f) \in \mathcal{OACT}((K, X, w, \triangleleft), (H, \mathrm{Ob}(G), \mathrm{Ob}(q), \triangleleft))$$

and that proves surjectivity. Let $(\phi, \psi) = (\phi', \psi') \in \mathcal{OCOV}(p, q)$, that implies $\mathrm{Ob}(\phi) = \mathrm{Ob}(\phi')$ which means $f = f'$ and surely $(\psi, f) = (\psi', f')$ so injectivity is also proved. $\square$

Remark 4.3.2. In fact the functors σ and ω give an equivalence of categories between $\mathcal{OACT}$ and $\mathcal{OCOV}$. This is outlined in [1, Proposition 4.3], and full details are given in [27, Section 5.3.4].

The Functor $\mathcal{OFIB} \to \mathcal{OACT}$

The factorization function $\theta \to \psi$ defined in Corollary 2.3.17 extends to a functor $\gamma : \mathcal{OFIB} \to \mathcal{OCOV}$ from the category of ordered fibrations to the category of ordered covering morphisms. γ is defined on objects of $\mathcal{OFIB}$ as

$$(\theta : G \to H)\gamma = \psi : G/_{\leqslant} \ker\theta \to H$$

and to define γ on arrows of $\mathcal{OFIB}$ Suppose given the following commutative diagram in $\mathcal{OFIB}(\theta, \theta')$

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\theta"'] & G' \arrow[d, "\theta'"] \\ H \arrow[r, "\beta"'] & H' \end{tikzcd}$$

where θ, θ' are ordered fibrations and α, β are ordered morphisms of groupoids. Notice that $(\ker\theta)\alpha \subseteq \ker\theta'$ so α gives $\alpha' : G/_{\leqslant} \ker\theta \to G'/_{\leqslant} \ker\theta'$. Now set $(\alpha, \beta)\gamma = (\alpha', \beta)$ to find the following commutative diagram in $\mathcal{OCOV}(\psi, \psi')$

$$\begin{tikzcd} G/_{\leqslant} \ker\theta \arrow[r, "\alpha'"] \arrow[d, "\psi"'] & G'/_{\leqslant} \ker\theta' \arrow[d, "\psi'"] \\ H \arrow[r, "\beta"'] & H' \end{tikzcd}$$

Let τ be the composite $\tau = \gamma\sigma : \mathcal{OFIB} \to \mathcal{OCOV} \to \mathcal{OACT}$ so that if $\theta : G \to H$ is an ordered fibration then $\theta\tau = (H, \mathrm{Ob}(G/_{\leqslant} \ker\theta), \mathrm{Ob}(\psi), \triangleleft)$ which means H acts on

$\mathrm{Ob}(G/_{\leqslant} \ker\theta)$.

4.4 The Maximum Enlargement Theorem

Theorem 2.3.16 gives a canonical factorization of an ordered groupoid morphism as a fibration followed by a star-injective functor. Ehresman's Maximum Enlargement Theorem [12] then provides a canonical factorization of a star-injective functor as a well-structured ordered embedding, called an enlargement, followed by a covering. In this section we give a short account of the Maximum Enlargement Theorem based solely on the notion of an ordered groupoid acting on a poset, this basis for a proof of the theorem was first set out in [16] and fully developed in [27].

We start with an ordered functor $\theta : G \to H$ and construct an ordered covering $\widetilde{H}$ of H using the immersion $\psi : G/_{\leqslant} \ker\theta \to H$. Then we prove the Maximum Enlargement Theorem that gives a factorization of ψ as an ordered enlargement followed by an ordered covering. After that we try to construct an ordered covering $\widetilde{H}$ using an ordered functor which is not necessarily an immersion. So we follow the same approach on θ and that leads to the same quotient $G/_{\leqslant} \ker\theta$ from the immersion ψ. The conclusion of this section is that any ordered functor $\theta : G \to H$ admits a canonical factorization as the composition of an ordered fibration followed by an ordered enlargement followed by an ordered covering.

Constructing a covering from an ordered immersion

Let $\theta : G \to H$ be an ordered functor of groupoids. We have the following factorization through $G/_{\leqslant} \ker\theta$:

$$\begin{tikzcd} G \arrow[dr, "\varpi"] \arrow[dd, "\theta"'] & \\ & G/_{\leqslant} \ker\theta \arrow[dl, "\psi"] \\ H & \end{tikzcd}$$

As we know ϖ is an ordered fibration and ψ is an ordered immersion as in Theorems 2.3.15 and 2.3.16. Denote $G/_{\leqslant} \ker\theta$ by U and let $[g] \in U([e],[f])$ and $h \in H$ such that $r(h) = [e]\psi$. Then U acts on the left of $\mathrm{Ob}(U)$ by $[g] \triangleright [f] = [e]$. Also U has right action on H given by $h \triangleleft [g] = h([g]\psi)$. The pullback $H \boxtimes \mathrm{Ob}(U)$ is defined as

$$H \boxtimes \mathrm{Ob}(U) = \{(s,[e]) \in H \times \mathrm{Ob}(U) : r(s) = [e]\psi\}$$

Define an equivalence relation on the pullback as follows

$$(s,[e]) \simeq_U (t,[f]) \text{ if and only if there exists } [g] \in U([e],[f]) \text{ such that } t = s([g]\psi)$$

The class of $(s,[e])$ is denoted by $s \otimes [e]$ and the quotient set is denoted by $H \otimes \mathrm{Ob}(U)$. A partial order relation is defined on $H \otimes \mathrm{Ob}(U)$ as follows

$$t \otimes [f] \leqslant s \otimes [e] \text{ if and only if there exists } [k] \leqslant [e] \text{ in } \mathrm{Ob}(U) \text{ with } [g] \in U([f],[k]) \text{ such that } t([g]\psi) \leqslant s$$

Equivalently,

$$t \otimes [f] \leqslant s \otimes [e] \text{ if and only if there exists } w \otimes [k] \in (H \otimes \mathrm{Ob}(U)) \text{ such that } w \otimes [k] = t \otimes [f] \text{ with } [k] \leqslant [e] \text{ in } \mathrm{Ob}(U) \text{ and } w \leqslant s \text{ in } H.$$

This partial ordering is well-defined since:

- If $t \otimes [f] \leqslant s \otimes [e]$, then clearly for any $t' \otimes [f'] = t \otimes [f]$ we have $t' \otimes [f'] = w \otimes [k]$ with $w \leqslant s$ and $[k] \leqslant [e]$ so we have $t' \otimes [f'] \leqslant s \otimes [e]$.
- Suppose that $s \otimes [e] = s' \otimes [e']$ with $t \otimes [f] \leqslant s \otimes [e]$. Then $t \otimes [f] \leqslant s' \otimes [e']$ for: As $s \otimes [e] = s' \otimes [e']$ there exists $[g] \in U([e],[e'])$ such that $s' = s([g]\psi)$. As $t \otimes [f] \leqslant s \otimes [e]$ there exists $[g'] \in U([f],[k])$ with $[k] \leqslant [e]$ such that $t([g']\psi) \leqslant s$. Put $[g_1] = [g']([k]|[g])$ with $r([k]|[g]) = [k']$. Then $[g_1] \in U([f],[k'])$ with $[k'] \leqslant [e']$, also $t([g_1]\psi) = t([g']\psi)([k]|[g])\psi \leqslant s([g]\psi) = s'$. Thus $t \otimes [f] \leqslant s' \otimes [e']$.

Lemma 4.4.1. $(H \otimes \mathrm{Ob}(U), \leqslant)$ *is a poset.*

Proof. (i) $t \otimes [f] = t \otimes [f]$, so $\leqslant$ is reflexive.

(ii) If $t \otimes [f] \leqslant s \otimes [e]$ and $s \otimes [e] \leqslant u \otimes [j]$ then there exists $[k] \leqslant [e]$ with $[g] \in U([f],[k])$ such that $t([g]\psi) \leqslant s$, and there exists $[k'] \leqslant [j]$ with $[g'] \in U([e],[k'])$ such that $s([g']\psi) \leqslant u$. As $[k] \leqslant [e]$ restrict $[g']$ to $[k]$ with $r([k]|[g']) = [i]$. Now let

$$p = t([g]\psi),\ q = s([g']\psi),\ [m] = [g]([k]|[g']) \text{ and } r = t([m]\psi), \text{ then}$$

$[i] \leqslant [k'] \leqslant [j]$ and $[m] \in U([f],[i])$. Further, $r = t([m]\psi) = t([g]([k]|[g']))\psi = t([g]\psi)([k]|[g'])\psi = p([k]|[g'])\psi$. But $([k]|[g']) \leqslant [g']$ and $p \leqslant s$ so we have $r \leqslant s([g']\psi) = q \leqslant u$ meaning $t([m]\psi) \leqslant u$ so $t \otimes [f] \leqslant u \otimes [j]$, so $\leqslant$ transitive.

(iii) Let $t \otimes [f] \leqslant t' \otimes [f']$ and $t' \otimes [f'] \leqslant t \otimes [f]$. Then there exists $[e] \leqslant [f']$ with $[g] \in U([f],[e])$ such that $t([g]\psi) \leqslant t'$. So $t \otimes [f] = t([g]\psi) \otimes [e]$. We can say that $[f] \leqslant [f']$ and $t \leqslant t'$.
There also exists $[k] \leqslant [f]$ with $[g'] \in U([f'],[k])$ such that $t'([g']\psi) \leqslant t$.

Now we have $[k] \leqslant [f] \leqslant [f']$ and $t'([g']\psi) \leqslant t \leqslant t'$. But $t'([g']\psi) = (d(t'[g']\psi)|t) = (d(t'[g']\psi)|t') = (d(t')|t') = t'$. And $(t'^{-1}t')([g']\psi) = t'^{-1}t'$ so $[g']\psi = [f']\psi$. Since ψ is an immersion we have $[g'] = [f']$ which makes $[k] = [f']$. So $[f'] \leqslant [f] \leqslant [f']$ and $[f] = [f']$. Also $t'([g']\psi) \leqslant t \leqslant t'$ gives $t' \leqslant t \leqslant t'$ implies $t = t'$. Therefore $t \otimes [f] = t' \otimes [f']$, so $\leqslant$ antisymmetric.

□

There is an action of H on the poset $(H \otimes \mathrm{Ob}(U))$ via $w : (H \otimes \mathrm{Ob}(U)) \to H_0$ mapping $s \otimes [e] \mapsto d(s)$. Let $h \in H$ and $s \otimes [e] \in (H \otimes \mathrm{Ob}(U))$, if $d(s) \leqslant r(h)$ then h acts on $s \otimes [e]$ as follows:

$$h \triangleright (s \otimes [e]) = (h|d(s))s \otimes [e]$$

w is order preserving since if $u \otimes [f] \leqslant s \otimes [e]$ then there exists $[k] \leqslant [e]$ with $[v] \in (U)([f],[k])$ such that $u([v]\psi) \leqslant s$ which makes $d(u) \leqslant d(s)$ so $(u \otimes [f])w \leqslant (s \otimes [e])w$. From this action, construct the action groupoid $(H \otimes \mathrm{Ob}(U)) \rtimes H = \{(s \otimes [e], h) : (s \otimes [e])w = d(h)\} = \{(s \otimes [e], h) : d(s) = d(h)\}$. Denote this action groupoid by $\widetilde{H}$.
If $(t \otimes [f], u), (s \otimes [e], v) \in \widetilde{H}$, then the composition is given by

$$(t \otimes [f], u)(s \otimes [e], v) = (t \otimes [f], uv)$$

whenever uv is defined in H and provided

$$u^{-1} \triangleright (t \otimes [f]) = (u^{-1}|d(t))t \otimes [f] = u^{-1}t \otimes [f] = s \otimes [e],$$

$d(t \otimes [f], u) = (t \otimes [f], d(u)) = (t \otimes [f], d(t))$ and $r(t \otimes [f], u) = (u^{-1}t \otimes [f], r(u))$
$(t \otimes [f], u)^{-1} = (u^{-1}t \otimes [f], u^{-1})$. So $\widetilde{H}$ is a groupoid with identities of the form $(t \otimes [f], d(t))$ and the ordering is defined as follows:

$$(t \otimes [f], u) \leqslant (s \otimes [e], v) \text{ if and only if } u \leqslant v \text{ and } t \otimes [f] \leqslant s \otimes [e].$$

The ordered action groupoid $\widetilde{H}$ is just the covering of H mentioned in the beginning of this section.

The Maximum Enlargement Theorem

Before we present the Maximum Enlargement Theorem, we need the definition of an *enlargement* of ordered groupoids.

Definition An enlargement of ordered groupoids is an ordered inclusion of a subgroupoid $A \hookrightarrow B$ such that

- $\mathrm{Ob}(A)$ is an order ideal in $\mathrm{Ob}(B)$: that is, if $x \in \mathrm{Ob}(B)$ and $y \in \mathrm{Ob}(A)$ with $x \leqslant y$, then $x \in \mathrm{Ob}(A)$,
- if $b \in B$ and $d(b), r(b) \in A$ then $b \in A$,
- if $e \in \mathrm{Ob}(B)$ then there exists $b \in B$ with $d(b) = e$ and $r(b) \in \mathrm{Ob}(A)$.

Theorem 4.4.2. *Let $\theta : G \to H$ be an ordered functor of groupoids, factorized through $U = G/_{\leqslant} \ker\theta$:*

$$\begin{tikzcd} G \arrow[r, "\varpi"] \arrow[d, "\theta"'] & U \arrow[dl, "\psi"] \\ H & \end{tikzcd}$$

Then:

1. *$\pi : \widetilde{H} \to H$ taking $(s \otimes [e], h) \mapsto h$ is an ordered covering of H.*
2. *$i : U \to \widetilde{H}$ taking $[g] \mapsto ([g]\psi \otimes r[g], [g]\psi)$ is an ordered embedding such that $i\pi = \psi$ and $\widetilde{H}$ is an enlargement of Ui.*
3. *Suppose that $j : U \to D$ is an ordered embedding and $\pi_1 : D \to H$ is an ordered covering such that $\psi = j\pi_1$, then there exists a unique ordered functor $\sigma : \widetilde{H} \to D$ such that $j = i\sigma$ and $\pi = \sigma\pi_1$.*

$$\begin{tikzcd} & \widetilde{H} \arrow[dr, bend left, "\pi"] \arrow[dd, "\sigma"] & \\ U \arrow[ur, bend left, "i"] \arrow[rr, "\psi"] \arrow[dr, bend right, "j"'] & & H \\ & D \arrow[ur, bend right, "\pi_1"'] & \end{tikzcd}$$

Proof. 1. Since π is the ordered projection from the action groupoid $(H \otimes \mathrm{Ob}(U)) \rtimes H$ to H, π is an ordered covering by Propositions 4.2.1 and 4.2.4.

2. It is clear that i is an ordered embedding and to show that $\widetilde{H}$ is an ordered enlargement of Ui we verify the three conditions of the definition above

   - To show that $\mathrm{Ob}(Ui)$ is an order ideal in $\mathrm{Ob}(\widetilde{H})$, suppose that $s \otimes [e] \leqslant [g]\psi \otimes r[g]$. Then $s \otimes [e] = s' \otimes [k]$ with $[k] \leqslant r[g] \in \mathrm{Ob}(U)$ and $s' \leqslant [g]\psi$. Since $[k]\psi = r(s')$ it follows that $s' = ([g]\psi|[k]\psi) = ([g]|[k])\psi$ and hence that $s \otimes [e] = ([g]|[k])\psi \otimes [k] \in \mathrm{Ob}(Ui)$.
   - Now suppose that $s \otimes [e] = d(s \otimes [e], h) = [g]\psi \otimes r[g]$ and that $h^{-1}s \otimes [e] = r(s \otimes [e], h) = [g']\psi \otimes r[g']$ for some $[g], [g'] \in U$. Hence there exists $[a] \in U([e], r[g])$ such that $[g]\psi = s([a]\psi)$ and $[b] \in U([e], r[g'])$ such that $[g']\psi = (h^{-1}s)([b]\psi)$. Then

$$h = s([b]\psi)([g']\psi)^{-1} = ([g]\psi)([a]\psi)^{-1}([b]\psi)([g']\psi)^{-1} = ([g][a]^{-1}[b][g']^{-1})\psi.$$

Now $i : [g][a]^{-1}[b][g']^{-1} \mapsto (h \otimes d[g'], h)$ but

$$h \otimes d[g'] = ([g][a]^{-1}[b][g']^{-1})\psi \otimes d[g'] = ([g][a]^{-1})\psi \otimes [e] = s \otimes [e]$$

and so $(s \otimes [e], h) \in Ui$.

- Now given $s \otimes [e] \in \mathrm{Ob}(\widetilde{H})$, the arrow $(s \otimes [e], s)$ has domain $s \otimes [e]$ and range $r(s) \otimes [e] = [e]\psi \otimes [e] \in \mathrm{Ob}(Ui)$.

3. Suppose that we are given a factorization of ψ as the composition $j\pi_1$ of an ordered embedding and an ordered covering. For $(s \otimes [e], h) \in \widetilde{H}$ we have $s \in cost_H([e]\psi) = cost_H([e]j\pi_1)$, and since π_1 is a covering there exists a unique $u \in cost_D([e]j)$ with $u\pi_1 = s$. Let $y = d(u)$ then $y\pi_1 = d(s) = d(h)$ and so there exists a unique $q \in D$ with $q^{-1} \in cost_D(y)$ and $q\pi_1 = h$. We define $(s \otimes [e], h)\sigma = q$.
This is well-defined since, if $s \otimes [e] = t \otimes [f]$ then there exist $[g] \in U([e], [f])$ with $s([g]\psi) = t$. Then the unique element of $cost_D([f]j)$ mapping to t is $u([g]j)$ and we obtain $y = d(u) = d(u([g]j))$ and hence q as before.
To show that σ is a functor, consider a composition $(s \otimes [e], h)(t \otimes [f], k) = (s \otimes [e], hk) \in \widetilde{H}$ such that $h^{-1}s \otimes [e] = t \otimes [f]$, with u, y, q defined as above, and $(s \otimes [e], h)\sigma = q$. There exists a unique v with $v^{-1} \in cost_D(r(q))$ with $v\pi_1 = k$. Then $(t \otimes [f], k)\sigma = v$ and $(s \otimes [e], hk)\sigma = qv$, and so σ is a functor. If $(s \otimes [e], h) \leqslant (m \otimes [f], l)$ then we have $h \leqslant l$ and we may assume that $[e] \leqslant [f]$ and $s \leqslant m$. We find a unique $n \in cost_D([f]j)$ with $n\pi_1 = m$. Then $r((n|[e]j)\pi_1) = [e]\psi$ and $(n|[e]j)\pi_1 \leqslant n\pi_1 = m$, hence $(n|[e]j) = u$. now there exists r with $r^{-1} \in cost_D(d(n))$ with $r\pi_1 = l$ and $(m \otimes [f], l)\sigma = r$. Since $u \leqslant n$ we have $y \leqslant d(n)$ and $(y|r)\pi_1 \leqslant l$. Hence $(y|r)\pi_1 = h$ and so $q = (y|r) \leqslant r$. Hence σ is an ordered functor.
To show that σ is unique with the stated properties, suppose that μ also possesses them. Then for $(s \otimes [e], h) \in \widetilde{H}$, we have

$$h = (s \otimes [e], h)\pi = (s \otimes [e], h)\sigma\pi_1 = (s \otimes [e], h)\mu\pi_1$$

and since π_1 is a covering, if μ and σ agree on the identity $h^{-1}s \otimes [e]$ then $(s \otimes [e], h)\sigma = (s \otimes [e], h)\mu$. But μ and σ agree on $\mathrm{Ob}(\widetilde{H})$, for since $i\sigma = j = i\mu$ we have that $\mu = \sigma$ on Ui, and then for any $z \in \mathrm{Ob}(\widetilde{H})$ we can join z to an $x \in (Ui)_0$ by some $a \in \widetilde{H}$. But then $a\mu = a\sigma$ and so $z\mu = z\sigma$.

□

Constructing a Covering from an Ordered Functor

In this discussion we try to construct the ordered covering $\widetilde{H}$ starting with the ordered functor $\theta : G \to H$ which is not necessarily an immersion. Let $\theta : G \to H$ be an ordered functor of groupoids. Then G acts on the right of H as $h \lhd g = h(g\theta)$ whenever $r(h) = (d(g))\theta$. Consider the pullback $H \boxtimes \mathrm{Ob}(G)$ as

$$H \boxtimes \mathrm{Ob}(G) = \{(s, e) \in H \times \mathrm{Ob}(G) : r(s) = e\theta\}.$$

Define an equivalence relation on the pullback as follows:

$$(s, e) \simeq_G (t, f) \text{ if there exists } g \in G(e, f) \text{ such that } t = s(g\theta).$$

The equivalence class of (s, e) is denoted by $s \otimes e$ and the quotient set is denoted by $H \otimes \mathrm{Ob}(G)$. An ordering on $H \otimes \mathrm{Ob}(G)$ is defined as follows:

$t \otimes f \leqslant s \otimes e$ if and only if there exists $k \leqslant e$ and there exists $g' \in G(f, k)$ such that $t(g'\theta) \leqslant s$.

This ordering relation is reflexive, transitive but not antisymmetric so $(H \otimes \mathrm{Ob}(G), \leqslant)$ is a preordered set. Hence, $H \otimes \mathrm{Ob}(G)$ has a canonical partially ordered quotient $(H \otimes \mathrm{Ob}(G))/Q$ by the equivalence relation

$(t \otimes f) \simeq_Q (s \otimes e)$ if and only if there exist $k, j \in \mathrm{Ob}(G)$ where $k \leqslant e, j \leqslant f$ and there exist $g' \in G(f, k), g \in G(e, j)$ such that $t(g'\theta) \leqslant s$ and $s(g\theta) \leqslant t$.

Remark 4.4.3. If we provide that g and g' above are in $\ker\theta$ then we have $t = s$. And in this case the pair (g, g') is a $\ker\theta$-nexus.

A class from the quotient $(H \otimes \mathrm{Ob}(G))/Q$ is denoted by $s \otimes [e]$. The ordering between these classes defined as follows:

$$t \otimes [f] \leqslant s \otimes [e] \text{ if and only if there exists } [k] \leqslant [e] \text{ and there exists}$$
$$[g] \in (G/_{\leqslant} \ker\theta)([f], [k]) \text{ such that } t([g]\psi) \leqslant s.$$

It is clear now that the poset $(H \otimes \mathrm{Ob}(G))/Q$,which is constructed from the ordered functor $\theta : G \to H$ is just the poset $H \otimes \mathrm{Ob}(U)$ obtained from the immersion $\psi : U \to H$. So the covering we aim to construct is just $(H \otimes \mathrm{Ob}(U)) \rtimes H$ constructed over ψ .

Corollary 4.4.4. *Any ordered functor $\theta : G \to H$ admits a canonical factorization as the composition of a fibration, followed by an ordered enlargement, followed by an ordered covering.*

$$G \to G/_{\leqslant} \ker\theta \to \widetilde{H} \to H.$$

4.5 The Factorization Theorem and The Derived Ordered Groupoid

In [30] Steinberg introduced a construction that he called the *derived ordered groupoid* $\mathsf{DER}(\phi)$ of a mapping $\phi : G \to H$ of ordered groupoids. Amongst its applications is the *Fibration Theorem* [30, Theorem 5.1] : any mapping of ordered groupoids can be factorized as the composition of an enlargement followed by a fibration.
The approach to $\mathsf{DER}(\phi)$ is based on a construction from topology, the *mapping cocylinder*, that is used to prove a similar theorem : every continuous map of topological spaces can be factorized as the composition of a homotopy equivalence followed by a fibration [33]. However, Steinberg's constructs the mapping cocylinder indirectly, as a semidirect product $\mathsf{DER}(\phi) \rtimes H$. We give a direct account of Steinberg's factorization, that follows and simplifies the formulation in [22].

We start with the groupoid $\mathsf{OGPD}(\mathfrak{I}, H)$ whose objects are the groupoid maps $\mathfrak{I} \to H$. However, any such map is completely determined by the image of the arrow ι of $\mathfrak{I}$, which is just an arrow of H. So we may identify the objects of $\mathsf{OGPD}(\mathfrak{I}, H)$ with the set of arrows of H. An arrow in $\mathsf{OGPD}(\mathfrak{I}, H)$ is then a natural transformation between maps $\mathfrak{I} \to H$. Identifying two such maps with arrows $h_0, h_1 \in H$, a natural transformation from h_0 to h_1 is then a pair of arrows (t_0, t_1) of H such that the square

$$\begin{tikzcd} \bullet \arrow[r, "h_0"] \arrow[d, "t_0"'] & \bullet \arrow[d, "t_1"] \\ \bullet \arrow[r, "h_1"'] & \bullet \end{tikzcd}$$

commutes. Now any three of the arrows in the square determines the fourth. We choose to suppress t_1 rewriting t_0 as t,the square above which is an arrow in $\mathsf{OGPD}(\mathfrak{I}, H)$ will be identified with the triple $[h_0, t, h_1]$ of arrows in H, which are subject to the condition that the composition $h_0^{-1} t h_1$ exists. Then we have

$$d[h_0, t, h_1] = h_0 \text{ and } r[h_0, t, h_1] = h_1,$$

and the composition of $[h_0, t, h_1]$ and $[h_1, u, s_1]$ is defined to be

$$[h_0, t, h_1][h_1, u, s_1] = [h_0, tu, s_1].$$

We then have an ordered functor $\epsilon_0 : \mathsf{OGPD}(\mathfrak{I}, H) \to H$ given by $\epsilon_0 : [h_0, t, h_1] \mapsto t$. It is clear that ϵ_0 is a fibration since for any $t \in \mathrm{star}_H([h, d(h), h]\epsilon_0)$ we have t is coverd by all elements of the form $[h_0, t, h_1]$ where $h_0^{-1} t h_1$ is defined in H.

Given $\phi : G \to H$ its mapping cocylinder M^ϕ is the pullback of the diagram

$$\begin{tikzcd} & \mathsf{OGPD}(\mathfrak{I}, H) \arrow[d, "\epsilon_0"] \\ G \arrow[r, "\phi"'] & H \end{tikzcd}$$

that is

$$M^\phi = \{(a, [h_0, a\phi, h_1]) : a \in G, h_0, h_1 \in H, h_0^{-1}(a\phi)h_1 \text{exists}\}.$$

We abbreviate the 4-tuple $(a, [h_0, a\phi, h_1])$ to the triple $\langle h_0, a, h_1 \rangle$ so that now

$$M^\phi = \{\langle h_0, a, h_1 \rangle : a \in G, h_0, h_1 \in H \text{and } h_0^{-1}(a\phi)h_1 \text{exists}\}$$

Then M^ϕ is an ordered groupoid with $\mathrm{Ob}(M^\phi)$ equal to the pullback

$$\{(e, h) \in \mathrm{Ob}(G) \times H : e\phi = d(h)\}.$$

We have

$$d\langle h_0, a, h_1 \rangle = (d(a), h_0) \text{ and } r\langle h_0, a, h_1 \rangle = (r(a), h_1),$$

and if $(r(a), h_1) = (d(b), k_0)$ then the composition of $\langle h_0, a, h_1 \rangle$ and $\langle k_0, b, k_1 \rangle$ is given by

$$\langle h_0, a, h_1 \rangle \langle k_0, b, k_1 \rangle = \langle h_0, ab, k_1 \rangle.$$

The ordering on arrows in M^ϕ is componentwise:

$$\langle h_0, a, h_1 \rangle \leqslant \langle k_0, b, k_1 \rangle \iff h_0 \leqslant k_0, a \leqslant b, \text{and } h_1 \leqslant k_1.$$

We can now prove Steinberg's Fibration Theorem [30]: our version follows that given in [22], but we show directly that p_ϕ has the covering homotopy property.

Theorem 4.5.1. *Let $\phi : G \to H$ be a morphism of ordered groupoids. Then ϕ admits a factorization*

$$\begin{tikzcd} G \arrow[r, "i_\phi"] \arrow[rr, bend right, "\phi"'] & M^\phi \arrow[r, "p_\phi"] & H \end{tikzcd}$$

where $i_\phi : g \mapsto \langle (d(g))\phi, g, (r(g))\phi \rangle$, M^ϕ is an enlargement of Gi_ϕ and $p_\phi : \langle h_0, a, h_1 \rangle \mapsto h_0^{-1}(a\phi)h_1$ is a strong ordered fibration.

Proof. It is clear that $\phi = i_\phi p_\phi$ and that i_ϕ, p_ϕ are order preserving. To show that i_ϕ and p_ϕ are morphisms of ordered groupoids, let ab be defined in G then

$$\begin{aligned}(ab)i_\phi &= \langle (d(a))\phi, ab, (r(b)\phi)\rangle \\ &= \langle (d(a)\phi), a, (r(a)\phi)\rangle\langle (d(b)\phi), b, (r(b)\phi)\rangle \\ &= ai_\phi bi_\phi,\end{aligned}$$

and

$$\begin{aligned}(\langle h_0, a, h_1\rangle\langle k_0, b, k_1\rangle)p_\phi &= \langle h_0, ab, k_1\rangle p_\phi \\ &= h_0^{-1}(ab)\phi k_1 \\ &= h_0^{-1}(a\phi)h_1 k_0^{-1}(b\phi)k_1 \\ &= \langle h_0, a, h_1\rangle p_\phi\langle k_0, b, k_1\rangle p_\phi,\end{aligned}$$

whenever $\langle h_0, a, h_1\rangle\langle k_0, b, k_1\rangle$ exists in M^ϕ.

Now, we show that p_ϕ is a strong ordered fibration, suppose we are given a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] & M^\phi \arrow[d, "p_\phi"] \\ A \times \mathfrak{I} \arrow[r, "F"'] & H \end{tikzcd}$$

For an arrow $a \in A$, let $af = \langle h_a, g_a, k_a\rangle \in M^\phi$ so that $(a,0)F = h_a^{-1}(g_a\phi)k_a$, and we have $(a,0)\widetilde{F} = af$ to guarantee the commutativity of the diagram. Suppose that $(r(a), \iota)F = l_a \in H$. Then

$$d(l_a) = d((r(a), \iota)F) = (r(a), 0)F = (r(a))fp_\phi = r(afp_\phi) = r(k_a).$$

Hence k_a and l_a are composable arrows in H, and we define

$$(r(a), \iota)\widetilde{F} = \langle k_a, g_a^{-1}g_a, k_a l_a\rangle.$$

so $(r(a), \iota)\widetilde{F}p_\phi = k_a^{-1}(g_a\phi)^{-1}(g_a\phi)k_a l_a = l_a$. Now we can define $\widetilde{F}$ on (a, ι) as follows:

$$\begin{aligned}(a, \iota)\widetilde{F} &= (a, 0)\widetilde{F}(r(a), \iota)\widetilde{F} \\ &= \langle h_a, g_a, k_a\rangle\langle k_a, r(g_a), k_a l_a\rangle \\ &= \langle h_a, g_a, k_a l_a\rangle\end{aligned}$$

Further, $(a, \iota)\widetilde{F}p_\phi = h_a^{-1}(g_a\phi)k_a l_a$. So $\widetilde{F}$ is an ordered lift of F.

Let $e \in \mathrm{Ob}(G)$ and let $h \in \mathrm{star}_H(e\phi)$. Then $\langle e\phi, e, h\rangle p_\phi = h$ with $d\langle e_\phi, e, h\rangle = (e, e\phi)$, showing that p_ϕ is star-surjective.

Now $Gi_\phi = \{\langle ((d(g))\phi, g, (r(g))\phi\rangle : g \in G\}$ is a subgroupoid of M^ϕ, with $\mathrm{Ob}(Gi_\phi) =$

$\{(e, e\phi) : e \in \text{Ob}(G)\}$. Suppose that $(x, h) \in \text{Ob}(M^\phi)$ and that $(x, h) \leqslant (e, e\phi)$. Then $x\phi = d(h)$, $x \leqslant e$ and $h \leqslant e\phi$. The last condition implies that $h \in H_0$, whence $h = x\phi$ and so $(x, h) \in \text{Ob}(Gi_\phi)$. We have shown that $\text{Ob}(Gi_\phi)$ is an order ideal in $\text{Ob}(M^\phi)$. Now suppose that $\langle h_0, a, h_1 \rangle \in M^\phi$ and that $(d(a), h_0) = (x, x\phi)$ and $(r(a), h_1) = (y, y\phi)$ with $x, y \in \text{Ob}(G)$. Then $\langle h_0, a, h_1 \rangle = \langle (d(a))\phi, a, (r(a))\phi \rangle \in Gi_\phi$, which establishes the second condition for an enlargement. Finally,take $(e, h) \in \text{Ob}(M^\phi)$. Then $d\langle h, e, e\phi \rangle = (e, h)$ and $r\langle h, e, e\phi \rangle = (e, e\phi) \in \text{Ob}(Gi_\phi)$. This concludes the proof that M^ϕ is an enlargement of Gi_ϕ. □

Corollary 4.5.2. *The kernel of the ordered fibration p_ϕ is equal to Steinberg's derived ordered groupoid* $\mathsf{DER}(\phi)$.

Proof. We have

$$\begin{aligned} \ker p_\phi &= \{\langle h_0, a, h_1 \rangle : h_0^{-1}(a\phi)h_1 \in H_0\} \\ &= \{\langle h_0, a, h_1 \rangle : h_0 = (a\phi)h_1\}. \end{aligned}$$

We can suppress mention of h_0 in elements of $\ker p_\phi$, and so write $\ker p_\phi = \{(a, h) : r(a\phi) = d(h)\}$. In this notation $(\ker p_\phi)_0 = \{(e, h) : e\phi = d(h)\}$ with $d(a, h) = (d(a), (a\phi)h)$ and $r(a, h) = (r(a), h)$. The composition of (a, h) and (b, k), defined when $(r(a), h) = (d(b), (b\phi)k)$, is then given by $(a, h)(b, k) = (ab, k)$. With due allowance for the change of notation required by Steinberg's use of left actions of groupoids, this is precisely his $\mathsf{DER}(\phi)$. □

Now H acts on $\mathsf{DER}(\phi)$: we have $w : (a, h) \mapsto r(h)$, so that $(a, h) \triangleleft h'$ is defined when hh' exists, and then $(a, h) \triangleleft h' = (a, hh')$.

Corollary 4.5.3. *The mapping cocylinder M^ϕ is isomorphic to the semidirect product $H \ltimes$* $\mathsf{DER}(\phi)$.

Proof. We define $\gamma : H \ltimes \mathsf{DER}(\phi) \to M^\phi$ by $(k, (g, h)) \mapsto \langle (g\phi)hk^{-1}, g, h \rangle$. Now the composition of $(k, (g, h))$ and $(l, (p, q))$ in $H \ltimes \mathsf{DER}(\phi)$ is defined if and only if $r(k) = d(l)$, $r(g) = d(p)$ and $h = (p\phi)ql^{-1}$, and then we have

$$(k, (g, h))(l, (p, q)) = (kl, (g, hl)(p, q)) = (kl, (gp, q)).$$

Then

$$\begin{aligned}(k,(g,h))\gamma(l,(p,q))\gamma &= \langle (g\phi)hk^{-1}, g, h\rangle\langle (p\phi)ql^{-1}, p, q\rangle \\ &= \langle (g\phi)hk^{-1}, g, h\rangle\langle h, p, q\rangle \\ &= \langle (g\phi)hk^{-1}, gp, q\rangle \\ &= \langle (g\phi)(p\phi)ql^{-1}k^{-1}, gp, q\rangle \\ &= (kl,(gp,q))\gamma.\end{aligned}$$

It follows that γ is a functor, and γ is then easily seen to be an isomorphism of ordered groupoids. $\square$

Remark 4.5.4. Considering the isomorphism $\gamma : H \ltimes \mathsf{DER}(\phi) \to M^\phi$ from the previous Corollary, there is another easier proof for p_ϕ being a strong fibration. The commutative triangle

$$\begin{tikzcd} M^\phi \arrow[rr, "\gamma^{-1}"] \arrow[dr, "p_\phi"'] & & H \ltimes \mathsf{DER}(\phi) \arrow[dl, "p"] \\ & H & \end{tikzcd}$$

and Proposition 4.2.1 shows directly that p_ϕ is a strong fibration.

Example 4.5.5. Take ϕ to be the inclusion $H_0 \hookrightarrow H$. Then $\mathsf{DER}(\phi)$ will be the analogue of the loop space ΩH of H, see [33]. Following this analogy, we then define

$$\Omega H = \{(d(h), h) : h \in H\}.$$

and so ΩH is identified as the trivial groupoid on the set of arrows of H, ordered as a partially ordered set by the ordering on H. This is the groupoid H_D of [30]. ΩH also occurs as the kernel of a fibration derived from $\epsilon_0 : \mathsf{OGPD}(\mathfrak{I}, H) \to H$. Let $\mathsf{OGPD}_*(\mathfrak{I}, H)$ be the kernel of ϵ_0, this is the analogue of the set of pointed maps from the unit interval (based at 0) to a pointed space (X, x_0). We have

$$\mathsf{OGPD}_*(\mathfrak{I}, H) = \{[h_0, e, h_1] : e \in H_0, h_0^{-1}eh_1 \text{exists}\}$$

and so $\mathsf{OGPD}_*(\mathfrak{I}, H)$ may be identified with the groupoid whose arrows are pairs of coinitial arrows in H, and a pair $(h_0, h_1) \in \mathsf{OGPD}_*(\mathfrak{I}, H)$ then satisfies $d(h_0, h_1) = h_0$ and $r(h_0, h_1) = h_1$ with composition $(h_0, h_1)(h_1, h_2) = (h_0, h_2)$. Then we have a fibration

$$\epsilon_1 : \mathsf{OGPD}_*(\mathfrak{I}, H) \to H \text{ mapping } (h_0, h_1) \mapsto h_0^{-1}h_1.$$

whose kernel is the subset $\{(h, h) : h \in H\}$ with $d(h, h) = h = r(h, h)$ and $(h, h)(h, h) = (h, h)$, and that is ΩH.

Proposition 4.5.6. *The splitting $q_\phi : M^\phi \to G$ of i_ϕ defined by $q_\phi : \langle h_0, a, h_1 \rangle \mapsto a$ is a fibration, and its restriction to $\mathsf{DER}(\phi)$ is a covering.*

Proof. Let $(e, h_0) \in \mathrm{Ob}(M^\phi)$ with $(e, h_0)q_\phi = e \in \mathrm{Ob}(G)$, and let $a \in \mathrm{star}_G(e)$ then a is covered by all elements of the form $\langle h_0, a, h \rangle$ in M^ϕ and $d\langle h_0, a, h \rangle = (e, h_0)$, so q_ϕ is a fibration and clearly ordered. To prove that the restriction $q_\phi|\mathsf{DER}(\phi)$ is a covering, it is sufficient to show that this restricted mapping has a discrete kernel. Using the notational changes introduced in the proof of Corollary 4.5.2 we have $q_\phi|\mathsf{DER}(\phi) : \mathsf{DER}(\phi) \to G$ given by $(a, h) \mapsto a$, and hence

$$\ker(q_\phi|\mathsf{DER}(\phi)) = \{(e, h) : e \in \mathrm{Ob}(G), e\phi = d(h)\} = \mathrm{Ob}(\mathsf{DER}(\phi)).$$

□

Remark 4.5.7. As we have seen, every ordered covering $\Gamma \to G$ has an associated action of G on the poset $\mathrm{Ob}(\Gamma)$. The action associated to $q_\phi|\mathsf{DER}(\phi)$ is given by

$$(d(g), h) \triangleleft g = (r(g), (g\phi)^{-1}h),$$

defined whenever $(d(g))\phi = d(h)$.

Chapter 5

The ordered derived module

This chapter is all about constructing the *ordered derived module* D_θ over the ordered functor $\theta : H \to G$ as a universal G-module, and involving it in the more complicated structure of the *derived chain complex*. Our definitions and constructions agree those given in [10] with ordering considered, and generalize some ideas worked out for inverse semigroups in [17].

There are seven sections in this chapter. The first one introduces the main concepts such as modules over ordered groupoids and the left cancellative category $C(G)$ of Loganathan [23]. The second section is about the *adjoint module* $\overrightarrow{\mathbb{Z}}G$ and the *augmentation module* $\overrightarrow{I}G$ and functors induced by such modules. The third section provides the definition of an ordered θ-derivation on an ordered functor θ . In the fourth section we construct the derived module D_θ over θ as a universal G-module. Section five is about the functor $\mathcal{D} : \mathcal{OG}^2 \to \mathcal{OMOD}$ obtained from the construction of the module D_θ and its right adjoint $\mathcal{X} : \mathcal{OMOD} \to \mathcal{OG}^2$. The main task of section six is to construct the derived chain complex functor $\Delta : \mathcal{OCRS} \to \mathcal{OCHN}$, while the seventh section is about a right adjoint of Δ, $\Theta : \mathcal{OCHN} \to \mathcal{OCRS}$.

5.1 Modules over Ordered Groupoids

In this section we discuss some known concepts like modules of ordered groupoids and the category $C(G)$ of Loganathan [23]: our descriptions follows that in [23] and [17].

Let G be an ordered groupoid. A G-module M consists of the following

- A collection of abelian groups M_x, one for each $x \in \mathrm{Ob}(G)$.
- For each arrow $g \in G(x, y)$ there is an isomorphism $\triangleleft g : M_x \to M_y$ satisfying the rules:

1. If g is the identity at $x \in \mathrm{Ob}(G)$, then $M_x \to M_x$ is the identity of the group M_x.
2. If g, h are composable in G, then the obtained isomorphisms preserve the composition $M_x \to M_y \to M_z$, that is $\lhd g \cdot \lhd h = \lhd(gh)$.

- If $x \leqslant y$ in $\mathrm{Ob}(G)$, then there is a morphism $\phi_x^y : M_y \to M_x$ obeying the rules:
  1. Since $x \leqslant x$ for all $x \in \mathrm{Ob}(G)$, we get an arrow $M_x \to M_x$ and this is the identity.
  2. If $x \leqslant y \leqslant z$ then $x \leqslant z$ and the composition is preserved here too, $\phi_y^z \phi_x^y = \phi_x^z$.
- if $e \leqslant d(g)$, then g has the restriction $(e \mid g)$ starting at e and finishing at f where $f \leqslant r(g)$ and we have the following commutative square:

$$\begin{tikzcd} M_{d(g)} \arrow[r, "\lhd g"] \arrow[d, "\phi_e^{d(g)}"'] & M_{r(g)} \arrow[d, "\phi_f^{r(g)}"] \\ M_e \arrow[r, "\lhd(e|g)"'] & M_f \end{tikzcd}$$

The structure of M_x as an abelian group and the ordering morphisms imply that $\sqcup_{x \in \mathrm{Ob}(G)} M_x = M$ is itself an ordered groupoid and this leads to the following alternative phrasing of the definition of M.

Definition Let G be an ordered groupoid. A G-module M is an ordered groupoid in which the components are abelian groups one for each object in G (a disjoint union of abelian groups), written as $M_x = \{m \in M : d(m) = r(m) = x\}$ and for each pair (m, g) with $m \in M, g \in G$ and $d(m) = d(g)$ there is an element $m \lhd g \in M_{r(g)}$ where the following axioms must be satisfied

OGM1 If g, h are composable in G and $m \lhd g$ is defined, then $(m \lhd g) \lhd h$ is defined and equal to $m \lhd (gh)$.

OGM2 If $m, n \in M_x$ with $d(g) = x$, then $(m + n) \lhd g = m \lhd g + n \lhd g \in M_{r(g)}$.
It follows that If 0_x is the identity of the group M_x , then $0_{d(g)} \lhd g = 0_{r(g)}$.

OGM3 If $m \in M_x$, then $m \lhd x = m$.

OGM4 If $m \lhd g$ and $n \lhd h$ are defined, and $m \leqslant n$ and $g \leqslant h$ then $m \lhd g \leqslant n \lhd h$.

Modules over ordered groupoids with the ordered morphism of modules form the category of ordered modules $\mathcal{OMOD}$.

Remark 5.1.1. The action of G on M described in the above definition is an example of an ordered groupoid action on an ordered groupoid, with considering the structure of M as a family of abelian groups over $\mathrm{Ob}(G)$.

Example 5.1.2. An important example of a G-module is $\overrightarrow{\mathbb{Z}}$ consisting of the constant family $(\overrightarrow{\mathbb{Z}})_x = \mathbb{Z}$ for any $x \in \mathrm{Ob}(G)$, and all mappings are the identity on $\mathbb{Z}$. For example, let $G = G_1 \sqcup G_0$ with $\phi : G_1 \to G_0$ where $g\phi \leqslant g$ for all $g \in G_1$

$$\begin{tikzcd} G_1 \arrow[d, "\phi"'] \\ G_0 \end{tikzcd}$$

Then for this G, $\overrightarrow{\mathbb{Z}}$ is

$$\begin{tikzcd} \mathbb{Z} \arrow[d, "\mathrm{id}"'] \\ \mathbb{Z} \end{tikzcd}$$

Regarding the action of the ordered groupoid G on the G-module, M, the semidirect product of G and M is the ordered action groupoid $G \ltimes M$. This semidirect product gives a functor

$$\ltimes : \mathcal{OMOD} \to \mathcal{OG}.$$

The Category $C(G)$

In the structure of a G-module in the ordered case, there are two different types of morphisms between the abelian groups M_x

(i) Ordering morphisms $M_e \to M_f$ when $e \geqslant f$ in $\mathrm{Ob}(G)$.

(ii) Action isomorphisms $M_{gg^{-1}} \to M_{g^{-1}g}$ induced by elements $g \in G$.

For dealing with both types in one construction we may replace G by the category $C(G)$ such that

1. Objects of $C(G)$ are the same as objects of G,
2. Arrows of $C(G)$ are pairs of the form (e, g) with $e \in \mathrm{Ob}(G)$, $g \in G$ and $e \geqslant d(g)$,

$$\begin{tikzcd} e \arrow[d] & \\ gg^{-1} \arrow[r, "g"'] & g^{-1}g \end{tikzcd}$$

3. $d(e, g) = e$ and $r(e, g) = r(g)$,

4. (e, g) composable with (f, h) if and only if $g^{-1}g = f$ and

$$(e, g)(f, h) = (e, (g \mid d(h))h).$$

$$\begin{tikzcd}
e \arrow[d] & & \\
gg^{-1} \arrow[r, "g"'] & f \arrow[d] & \\
& hh^{-1} \arrow[r, "h"'] & h^{-1}h
\end{tikzcd}$$

This construction is due to Loganathan [23].

Remark 5.1.3. $C(G)$ is a category and is not usually a groupoid.

Lemma 5.1.4. *$C(G)$ is a left cancellative category, meaning if $(e, g)(f, h) = (e, g)(k, p)$ then $(f, h) = (k, p)$.*

Proof. $(e, (g \mid d(h))h) = (e, (g \mid d(p))p)$ this equality implies that $(g \mid d(h))h = (g \mid d(p))p$ which means $d(g \mid d(h)) = d(g \mid d(p))$ and $d(g \mid d(h)) \mid g = d(g \mid d(p)) \mid g$ but we know that $d(g \mid d(h)) \mid g = g \mid d(h)$ and that leads to $h = p$, thus $(f, h) = (k, p)$. $\square$

Remark 5.1.5. G can be regarded as a subcategory of $C(G)$ by defining the injective functor $G \to C(G)$ as $g \mapsto (d(g), g)$. The poset $\mathrm{Ob}(G)$ is also a subcategory of $C(G)$ and each arrow of $C(G)$ has a unique decomposition

$$(e, g) = (e, d(g))(d(g), g)$$

with the first factor in $\mathrm{Ob}(G)$ and the second in G. This is an example of a Zappa-Szép product (or bicrossed product) of categories, see [2, 13].

An ordered G-module is now a functor $C(G) \to \mathcal{AB}$ to the category of abelian groups mapping:

- Objects $x \in \mathrm{Ob}(G)$ to the abelian groups M_x.
- Arrows (e, f) to the ordering morphisms of groups $M_e \to M_f$,when $e \geqslant f \in \mathrm{Ob}(G)$.
- Arrows (gg^{-1}, g) to actions $M_{gg^{-1}} \to M_{g^{-1}g}$.
- Arrows $(e, g) = (e, gg^{-1})(gg^{-1}, g)$ to the composition $M_e \to M_{gg^{-1}} \to M_{g^{-1}g}$.

5.2 Adjoint Module and Augmentation Module

In this section we provide the definitions of the adjoint module $\overrightarrow{\mathbb{Z}}G$ and the augmentation module $\overrightarrow{I}G$ over the ordered groupoid G that follow definitions of [9, 10] in the unordered case, and generalize to ordered groupiods the definitions of [23].

Definition For an ordered groupoid G the adjoint module $\overrightarrow{\mathbb{Z}}G$ is a G-module in which, for an identity $x \in \mathrm{Ob}(G)$, the abelian group $(\overrightarrow{\mathbb{Z}}G)_x$ is the free abelian group with basis all arrows in G that end at x. Thus an element of $(\overrightarrow{\mathbb{Z}}G)_x$ has uniquely the form of a finite sum $\sum a_p p$ with $a_p \in \mathbb{Z}$ and $r(p) = x$.

To explain the action of G on $\overrightarrow{\mathbb{Z}}G$ let $g \in G(x, y)$ and $\sum a_p p \in (\overrightarrow{\mathbb{Z}}G)_x$, then $\sum a_p p \triangleleft g = \sum a_p(pg) \in (\overrightarrow{\mathbb{Z}}G)_y$. If $e \geqslant f \in \mathrm{Ob}(G)$ and u is a basis element in $(\overrightarrow{\mathbb{Z}}G)_e$, then $(u \mid f)$ is a basis element in $(\overrightarrow{\mathbb{Z}}G)_f$ and this corestriction gives us a morphism $(\overrightarrow{\mathbb{Z}}G)_e \to (\overrightarrow{\mathbb{Z}}G)_f$. This construction defines a functor

$$\overrightarrow{\mathbb{Z}}(-) : \mathcal{OG} \to \mathcal{OMOD}$$

Definition The map $\varepsilon : \overrightarrow{\mathbb{Z}}G \to \overrightarrow{\mathbb{Z}}$ given by $\sum a_p p \to \sum a_p$ is a morphism of G-modules called the *augmentation* map, and its kernel $\overrightarrow{I}G$ is called the *augmentation module* of G.

Any ordered functor of groupoids $\theta : H \to G$ induces an ordered morphism $\overrightarrow{\mathbb{Z}}H \to \overrightarrow{\mathbb{Z}}G$ over θ which maps $\overrightarrow{I}H$ to $\overrightarrow{I}G$. The augmentation module also defines a functor

$$\overrightarrow{I} : \mathcal{OG} \to \mathcal{OMOD}$$

Lemma 5.2.1. *Let G be an ordered groupoid and $x \in \mathrm{Ob}(G)$, then $(\overrightarrow{I}G)_x$, as a free abelian group, has a $\mathbb{Z}$- basis consisting of all $g - 1_x$, for g a non-identity element of G with target x.*

Proof. Let $w = \sum a_g g$ be an element of the abelian group $(\overrightarrow{I}G)_x$, then $\sum a_g = 0$ and $w = \sum a_g g - \sum a_g = \sum a_g(g - 1_x)$. □

5.3 Ordered θ-derivations

In this section, we use the definition of θ-derivation from [9, 10]. After that we consider the ordering to find out the necessary and sufficient conditions for a function to be an *ordered θ-derivation*. We prove that the functor $\ltimes : \mathcal{OMOD} \to \mathcal{OG}$ is a right adjoint of $\overrightarrow{I} : \mathcal{OG} \to \mathcal{OMOD}$.

Definition Let $\theta : H \to G$ be a morphism of groupoids and M a G-module. A function $f : H \to M$ is called a θ-derivation if it maps $H(p,q)$ to $M_{q\theta}$ and satisfies

$$(xy)f = (xf) \triangleleft y\theta + yf$$

whenever xy is defined in H. In particular, an id_G-derivation is called a *derivation*.

Example 5.3.1. Let G be a groupoid and let $\psi : G \to \overrightarrow{I}G$ be given by $g \mapsto g - 1_{r(g)}$. We have

$$\begin{aligned} g\psi \triangleleft h + h\psi &= (g - 1_{r(g)}) \triangleleft h + h - 1_{r(h)} \\ &= gh - h + h - 1_{r(h)} = gh - 1_{r(h)} \\ &= (gh)\psi \end{aligned}$$

and certainly $g - 1_{r(g)} \in (\overrightarrow{I}G)_{r(g)}$ which makes ψ a derivation.

Moreover, ψ has an important universal property that is, if $f : G \to N$ is a derivation to a G-module, N, then there is a unique morphism of G-modules $f' : \overrightarrow{I}G \to N$ such that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\psi"'] & N \\ \overrightarrow{I}G \arrow[ur, dashed, "f'"'] & \end{tikzcd}$$

where $g - 1_{r(g)} \mapsto gf$.

To define the ordered θ-derivation $f : H \to M$ over the ordered functor $\theta : H \to G$, we obtain some properties which are necessary and sufficient for f to be such a derivation. Let $F : H \to G \ltimes M$ be an ordered functor that has the form $hF = (h\theta, hf)$. Then we have the following facts:

- Since F is a morphism of groupoids we have

  $$(h_1h_2)F = (h_1)F(h_2)F = (h_1\theta, h_1f)(h_2\theta, h_2f) = (h_1\theta h_2\theta, h_1f \triangleleft h_2\theta + h_2f)\,,$$

  which means f must satisfy the property $(h_1h_2)f = h_1f \triangleleft h_2\theta + h_2f$.
- Moreover, since F is ordered, if $h_1 \leqslant h_2$, then $h_1F \leqslant h_2F$ that is $(h_1\theta, h_1f) \leqslant (h_2\theta, h_2f)$ we assumed that θ is ordered so $h_1\theta \leqslant h_2\theta$ and f must satisfy $h_1f \leqslant h_2f$ that is $h_1f = (h_2f)\phi$ where $\phi : M_{r(h_2\theta)} \to M_{r(h_1\theta)}$.
- We also have $hf \in M_{r(h\theta)}$ satisfied according to the structure of $G \ltimes M$.

So these three properties are necessary for f to be an ordered θ-derivation and for F to be an ordered functor. So we make the following definition. Let M be a G-module and

$\theta : H \to G$ an ordered functor. Then an ordered θ–derivation $f : H \to M$ is a function satisfying the three properties:

- $hf \in M_{r(h\theta)}$,
- $(h_1h_2)f = h_1f \lhd h_2\theta + h_2f$,
- if $h_1 \leqslant h_2$, then $h_1f \leqslant h_2f$ so $h_1f = (h_2f)\phi$ where $\phi : M_{r(h_2\theta)} \to M_{r(h_1\theta)}$.

If we now define $F : H \to G \ltimes M$ as $hF = (h\theta, hf)$ we then have $(h_1h_2)F = (h_1h_2\theta, h_1h_2f) = (h_1\theta h_2\theta, h_1f \lhd h_2\theta + h_2f) = (h_1\theta, h_1f)(h_2\theta, h_2f)$. Further, if $h_1 \leqslant h_2$ then as we assumed we have $h_1\theta \leqslant h_2\theta$ and $h_1f \leqslant h_2f$ which implies that $(h_1\theta, h_1f) \leqslant (h_2\theta, h_2f)$ that is $h_1F \leqslant h_2F$ so F is an ordered functor of groupoids.

Remark 5.3.2. The above three conditions can be amalgamated into one condition by acting with $C(G)$ instead of G, as follows: if $h_1, h_2 \in H$ with $h_1^{-1}h_1 \geqslant h_2h_2^{-1}$, then we have the composition $(h_1 \mid h_2h_2^{-1})h_2$ and $f : H \to M$ must satisfy the following condition to be a θ-derivation :

$$((h_1 \mid h_2h_2^{-1})h_2)f = (h_1 \mid h_2h_2^{-1})f \lhd h_2\theta + h_2f,$$

but we know that $h_1 \geqslant (h_1 \mid h_2h_2^{-1})$ so $(h_1 \mid h_2h_2^{-1})f = h_1f\phi$ where $\phi : M_{(h_1^{-1}h_1)\theta} \to M_{(h_2h_2^{-1})\theta}$ is the ordering morphism. Now, we have

$$\begin{aligned}((h_1 \mid h_2h_2^{-1})h_2)f &= h_1f\phi \lhd h_2\theta + h_2f\\ &= h_1f \lhd ((h_1^{-1}h_1)\theta, (h_2h_2^{-1})\theta) \lhd ((h_2h_2^{-1})\theta, h_2\theta) + h_2f,\end{aligned}$$

and the necessary and sufficient condition is in the following form:

$$((h_1 \mid h_2h_2^{-1})h_2)f = h_1f \lhd ((h_1^{-1}h_1)\theta, h_2\theta) + h_2f \tag{5.3.1}$$

In the case $h_1^{-1}h_1 = h_2h_2^{-1}$, we have

$$\begin{aligned}(h_1h_2)f &= h_1f \lhd ((h_2h_2^{-1}\theta), h_2\theta) + h_2f\\ &= h_1f \lhd h_2\theta + h_2f.\end{aligned}$$

And if we take the case $h_2 = d(h_2) \leqslant h_1^{-1}h_1$, we then have

$$\begin{aligned}(h_1 \mid d(h_2))f &= h_1f \lhd ((h_1^{-1}h_1)\theta, (h_2h_2^{-1})\theta) + (d(h_2))f\\ &= (h_1f)\phi.\end{aligned}$$

Proposition 5.3.3. *The functor $\ltimes : \mathcal{OMOD} \to \mathcal{OG}$ is a right adjoint of $\overrightarrow{I} : \mathcal{OG} \to \mathcal{OMOD}$.*

Proof. We aim to show a natural bijection

$$\mathcal{OMOD}(\overrightarrow{I}H, M) \cong \mathcal{OG}(H, G \ltimes M).$$

So if we start with an element $F \in \mathcal{OG}(H, G \ltimes M)$, we need to find a corresponding element $\tilde{f} \in \mathcal{OMOD}(\overrightarrow{I}H, M)$. A morphism $F : H \to G \ltimes M$ is of the form $h \mapsto (h\theta, hf)$ where $\theta : H \to G$ is an ordered morphism of groupoids, and $f : H \to M$ is an ordered θ-derivation. As we shall see later in the next section, the map $\psi : H \to \overrightarrow{I}H$, given by $h\psi = h - 1_q$ for $h \in H(p, q)$, is a universal ordered derivation. That means any θ-derivation $f : H \to M$ is uniquely of the form $f = \psi\tilde{f}$ where $\tilde{f} : \overrightarrow{I}H \to M$ is an ordered morphism of modules over θ. □

5.4 The Ordered Derived Module D_θ

We dedicate this section to constructing the ordered derived module D_θ over the ordered functor $\theta : H \to G$ with the ordered θ-derivation $\delta : H \to D_\theta$, and we end it by proving the following universal property of D_θ:

If $\alpha : H \to M$ is a θ-derivation to the G-module M, then there is a unique morphism of G-modules $\beta : D_\theta \to M$ such that $\alpha = \delta\beta$.

Let $\theta : H \to G$ be an ordered functor, we aim to construct the *derived module* D_θ with a θ-derivation $\delta : H \to D_\theta$. First, construct a G-module F as follows: for each $e \in \mathrm{Ob}(G)$, let F_e be the free abelian group generated by all pairs

$$\{(h, g) : (h^{-1}h)\theta \geqslant gg^{-1}; g^{-1}g = e\}.$$

An action by $C(G)$ on F is defined as follows:

$$(h, g) \triangleleft (e, t) = (h, (g \mid d(t))t)\,, \text{ mapping } F_{g^{-1}g} \to F_{t^{-1}t}.$$

Now let f be the ordered function $f : H \to F$ mapping $h \mapsto (h, (h^{-1}h)\theta)$. Then f is a θ-derivation if and only if it satisfies the condition 5.3.1, that is if $h_1, h_2 \in H$ with $h_1^{-1}h_1 \geqslant h_2h_2^{-1}$ then we want f to satisfy the following

$$\begin{aligned}((h_1 \mid h_2h_2^{-1})h_2)f &= h_1f \triangleleft ((h_1^{-1}h_1)\theta, h_2\theta) + h_2f \\ &= (h_1, (h_1^{-1}h_1)\theta) \triangleleft ((h_1^{-1}h_1)\theta, h_2\theta) + (h_2, (h_2^{-1}h_2)\theta)\end{aligned}$$

That is

$$((h_1 \mid h_2h_2^{-1})h_2, (h_2^{-1}h_2)\theta) = (h_1, h_2\theta) + (h_2, (h_2^{-1}h_2)\theta) \tag{5.4.1}$$

Equation 5.4.1 can not be true in general since it is a relation between basis elements in F, but we shall impose relations on F to make 5.4.1 true.

Start by acting on both sides of 5.4.1 by any $g \in G$ where $gg^{-1} \leqslant (h_2^{-1}h_2)\theta$, then we have

$$((h_1 \mid h_2h_2^{-1})h_2, g) = (h_1, (h_2\theta \mid gg^{-1})g) + (h_2, g)\,,$$

and take K_e to be subgroup of F_e generated by all elements

$$((h_1 \mid h_2h_2^{-1})h_2, g) - (h_1, (h_2\theta \mid gg^{-1})g) - (h_2, g)\,.$$

Now, define $(D_\theta)_e$ to be the quotient group F_e/K_e, and let $k : F \to D_\theta$ be the quotient mapping. Then we have

$$H \xrightarrow{\ f\ } F \xrightarrow{\ k\ } D_\theta$$

and $\delta = fk : H \to D_\theta$ mapping $h \mapsto (h, (h^{-1}h)\theta)$.

Proposition 5.4.1. *Let $\theta : H \to G$ be an ordered functor with derived module D_θ and $\theta-$ derivation, $\delta : H \to D_\theta$. If $\alpha : H \to M$ is a θ-derivation to the G-module M, then there is a unique morphism of G-modules $\beta : D_\theta \to M$ that makes the diagram*

$$\begin{tikzcd} H \arrow[r, "\delta"] \arrow[d, "\alpha"'] & D_\theta \arrow[dl, "\beta"] \\ M & \end{tikzcd}$$

commute.

Proof. To define this β we shall go through the following steps

1. Defining β on $Im(\delta)$ by assuming the commutativity of the diagram. Let $h \in H$, we know that $h\delta = (h, (h^{-1}h)\theta)$ and the only way that guarantees the commutativity of the diagram is to put

$$(h, (h^{-1}h)\theta)\beta = h\alpha$$

2. Defining β as a morphism of abelian groups:

   - define β on the generators of D_θ in a way consistent with the intention that β is going to be a morphism of $C(G)$-modules, we then have

$$\begin{aligned}(h, g)\beta &= ((h, (h^{-1}h)\theta) \triangleleft ((h^{-1}h)\theta, g))\beta \\ &= (h, (h^{-1}h)\theta)\beta \triangleleft ((h^{-1}h)\theta, g) \\ &= h\alpha \triangleleft ((h^{-1}h)\theta, g)\end{aligned}$$

- we also need to check that $((h_1 \mid h_2h_2^{-1})h_2, g)\beta = (h_1, (h_2\theta \mid gg^{-1})g)\beta + (h_2, g)\beta$

$$\begin{aligned}
R.H.S. &= h_1\alpha \triangleleft ((h_1^{-1}h_1)\theta, (h_2\theta \mid gg^{-1})g) + h_2\alpha \triangleleft ((h_2^{-1}h_2)\theta, g) \\
&= h_1\alpha \triangleleft ((h_1^{-1}h_1)\theta, (h_2h_2^{-1})\theta) \triangleleft ((h_2h_2^{-1})\theta, (h_2\theta \mid gg^{-1})g) + h_2\alpha \triangleleft ((h_2^{-1}h_2)\theta, g) \\
&= (h_1 \mid h_2h_2^{-1})\alpha \triangleleft ((h_2h_2^{-1})\theta, (h_2\theta \mid gg^{-1})g) + h_2\alpha \triangleleft ((h_2^{-1}h_2)\theta, g)
\end{aligned}$$

On the other side we have

$$\begin{aligned}
L.H.S. &= ((h_1 \mid h_2h_2^{-1})h_2)\alpha \triangleleft ((h_2^{-1}h_2)\theta, g) \\
&= [(h_1 \mid h_2h_2^{-1})\alpha \triangleleft (h_2h_2^{-1}\theta, h_2\theta) + h_2\alpha] \triangleleft ((h_2^{-1}h_2)\theta, g) \\
&= (h_1 \mid h_2h_2^{-1})\alpha \triangleleft (h_2h_2^{-1}\theta, h_2\theta) \triangleleft ((h_2^{-1}h_2)\theta, g) + h_2\alpha \triangleleft ((h_2^{-1}h_2)\theta, g) \\
&= (h_1 \mid h_2h_2^{-1})\alpha \triangleleft ((h_2h_2^{-1})\theta, (h_2\theta \mid gg^{-1})g) + h_2\alpha \triangleleft ((h_2^{-1}h_2)\theta, g)
\end{aligned}$$

The construction of β, defined on the free generators and preserving the above relation, ensures that β is an abelian group morphism.

3. Verifying that β is a morphism of $C(G)$-modules, and for that let $(h, g) \in D_\theta$ and $(kk^{-1}, k) \in C(G)$ with $g^{-1}g \geqslant kk^{-1}$, then

$$\begin{aligned}
((h, g) \triangleleft (kk^{-1}, k))\beta &= (h, (g \mid kk^{-1})k)\beta \\
&= h\alpha \triangleleft ((h^{-1}h)\theta, (g \mid kk^{-1})k) \\
&= h\alpha \triangleleft ((h^{-1}h)\theta, g) \triangleleft (kk^{-1}, k) \\
&= (h, g)\beta \triangleleft (kk^{-1}, k)
\end{aligned}$$

□

Proposition 5.4.2. *For an ordered groupoid G, the augmentation module $\overrightarrow{I}G$ is the ordered derived module of the identity $G \to G$.*

Proof. $\overrightarrow{I}G$ is a G- module according to its definition, so all we need is to define a derivation $G \to \overrightarrow{I}G$ and prove the universal property of that derivation. We have proved this for the unordered case in Example 5.3.1, so we now consider the ordering and define $\psi : G \to \overrightarrow{I}G$ such that $g \mapsto g - 1_{r(g)}$ and let $g, h \in G$ with $g^{-1}g \geqslant hh^{-1}$. We want ψ to satisfy the condition

$$((g \mid hh^{-1})h)\psi = (g \mid hh^{-1})\psi \triangleleft (g^{-1}g, h) + h\psi$$

$$\begin{aligned} R.H.S. &= (g \mid hh^{-1} - 1_{d(h)}) \triangleleft h + h - 1_{r(h)} \\ &= (g \mid hh^{-1})h - h + h - 1_{r(h)} \\ &= (g \mid hh^{-1})h - 1_{r(h)} \\ &= L.H.S. \end{aligned}$$

To prove the universal property, let $\alpha : G \to M$ be an ordered derivation to the G-module M, and try to define a unique morphism of G-modules $\beta : \overrightarrow{I}G \to M$ such that the diagram commutes

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, "\alpha"'] & \overrightarrow{I}G \arrow[dl, "\beta"] \\ M & \end{tikzcd}$$

The only way to guarantee commutativity of the diagram is to define β on $Im(\psi)$ as follows:

$$(g\psi)\beta = (g - 1_{r(g)})\beta = g\alpha$$

Since we defined β on the elements of a basis for $\overrightarrow{I}G$ as a free abelian group, we surely have β as a group morphism and the only thing needed is to check if β a morphism of G-modules, for that let $h \in G$ such that gh exists then we have

$$\begin{aligned} (g - 1_{r(g)})\beta \triangleleft h &= g\alpha \triangleleft h \\ &= (gh)\alpha - h\alpha \\ &= (gh - 1_{r(h)})\beta - (h - 1_{r(h)})\beta \\ &= (gh - 1_{r(h)} - h + 1_{r(h)})\beta \\ &= (gh - h)\beta \\ &= ((g - 1_{r(g)}) \triangleleft h)\beta. \end{aligned}$$

$\square$

5.5 The functor $\mathcal{D}$ and the right adjoint $\mathcal{X}$

Let $\mathcal{OG}^2$ be the category of ordered morphisms of ordered groupoids: its objects are ordered functors $G \to H$ between ordered groupoids, and the arrows are the obvious commutative squares. The construction of the ordered derived module D_θ yields a functor $\mathcal{D} : \mathcal{OG}^2 \to \mathcal{OMOD}$ from the category of morphisms of ordered groupoids to the category of modules of ordered groupoids. In this section we construct a functor $\mathcal{X} : \mathcal{OMOD} \to \mathcal{OG}^2$ and prove that it is a right adjoint of $\mathcal{D}$.

$\mathcal{D}$ is defined on objects as $(\theta : H \to G)\mathcal{D} = D_\theta$, and on arrows $\mathcal{D}$ should map the following square, as an arrow in $\mathcal{OG}^2$,

$$
\begin{tikzcd}
H \arrow[r, "f"] \arrow[d, "\theta"'] & F \arrow[d, "\phi"] \\
G \arrow[r, "k"'] & K
\end{tikzcd}
$$

to the morphism of ordered modules $X : D_\theta \to D_\phi$. To construct such a morphism, consider the following commutative diagram

$$
\begin{tikzcd}
 & H \arrow[r, "f"] \arrow[dl, "\delta_\theta"'] \arrow[drr, dashed, "\eta"] & F \arrow[dr, "\delta_\phi"] & \\
D_\theta \arrow[rrr, bend right, "X"'] & & & D_\phi
\end{tikzcd}
$$

where f is the ordered morphism of groupoids from the previous square, δ_θ is the universal θ-derivation and δ_ϕ is the universal ϕ-derivation, if we now define η as an $f\phi$-derivation then X definitely exists by using the universal property of D_θ.
From the commutativity of the diagram we can define η as follows:

$$h\eta = hf\delta_\phi$$

and now we show that η is an $f\phi$-derivation

$$
\begin{aligned}
(h_1h_2)\eta &= [(h_1h_2)f]\delta_\phi \\
&= [(h_1f)(h_2f)]\delta_\phi \\
&= (h_1f\delta_\phi) \triangleleft (h_2f\phi) + h_2f\delta_\phi \\
&= h_1\eta \triangleleft h_2f\phi + h_2\eta
\end{aligned}
$$

Thus, we have defined $\mathcal{D}$ on arrows as it should be, and the following proposition gives a right adjoint of $\mathcal{D}$.

Proposition 5.5.1. *$\mathcal{D}$ has a right adjoint $\mathcal{X} : \mathcal{OMOD} \to \mathcal{OG}^2$ such that $M \mapsto (p : G \ltimes M \to G)$.*

Proof. We need to show that the mapping of sets $\mathcal{OG}^2(\theta, p) \longrightarrow \mathcal{OMOD}(D_\theta, M)$ is a bijection. For this purpose, assume at the beginning that we have $\alpha : D_\theta \to M \in \mathcal{OMOD}(D_\theta, M)$. Since D_θ is the ordered derived module of G by θ, then surely we have the θ-derivation $\delta : H \to D_\theta$, and if we take f to be the composition $\delta\alpha : H \to M$ then f is an ordered θ-derivation and that enables us to define the ordered functor $F : H \to G \ltimes M$

where $F : h \mapsto (h\theta, hf)$. This F makes the following diagram commute

$$\begin{tikzcd}
H \arrow[r, "F"] \arrow[d, "\theta"'] & G \ltimes M \arrow[d, "p"] \\
G \arrow[r, "i"'] & G
\end{tikzcd}$$

where i is the identity mapping and p is the projection to G. This commutative square is actually an element in the set $\mathcal{OG}^2(\theta, p)$.
Now, assume that we have the commutative square

$$\begin{tikzcd}
H \arrow[r, "F"] \arrow[d, "\theta"'] & G \ltimes M \arrow[d, "p"] \\
G \arrow[r, "i"'] & G
\end{tikzcd}$$

as an element in the set $\mathcal{OG}^2(\theta, p)$, then $F : h \mapsto (h\theta, hf)$ is an ordered functor and $f : H \to M$ is an ordered θ-derivation. By the universal property of D_θ, there is a unique ordered morphism of G-modules $\alpha : D_\theta \to M$ which is an element in the set $\mathcal{OMOD}(D_\theta, M)$.

Finally, if we define two functions S and T between the two sets we started with as $S : \mathcal{OMOD}(D_\theta, M) \to \mathcal{OG}^2(\theta, p)$ such that $\alpha \mapsto (F, i)$ and $T : \mathcal{OG}^2(\theta, p) \to \mathcal{OMOD}(D_\theta, M)$ such that $(F, i) \mapsto \alpha$ then obviously, $ST : \alpha \to \alpha$ and $TS : (F, i) \to (F, i)$ are the identities. □

5.6 Crossed complexes and chain complexes over ordered groupoids

In this section we consider the ordering and follow the approach of [9, 10] to explain the structures of crossed complex and chain complex over ordered groupoids. Then we involve the adjoint module and the derived module to construct the derived chain complex functor $\Delta : \mathcal{OCRS} \to \mathcal{OCHN}$, in the ordered case, from the category of ordered crossed complexes to the category of ordered chain complexes.

The following definition of ordered crossed modules is needed in the beginning of our discussion.

Definition A crossed module over an ordered groupoid G is given by an ordered groupoid M and an ordered functor $\mu : M \to G$ which is the identity on objects and satisfies:

- M is a family of groups $\{M_x\}_{x \in \mathrm{Ob}(G)}$.

- μ is given by a family of group morphisms $\{\mu_x : M_x \to G_x\}_{x \in \mathrm{Ob}(G)}$.
- G acts on M.

These data are to satisfy two axioms:

CM1 μ preserves the action, that is $(m \triangleleft g)\mu = g^{-1}(m\mu)g = m\mu \triangleleft g$.

CM2 For any $m \in M_x$, $m \triangleleft n\mu = n^{-1}mn$ for all $n \in M_x$.

The category of ordered crossed complexes

A crossed complex C over an ordered groupoid C_1 is a sequence

$$\ldots \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \ldots\ldots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

giving by the following sets of data:

1. For $n \geqslant 3$, C_n is a C_1–module.
2. $\delta_2 : C_2 \to C_1$ is an ordered crossed module over C_1.

These sets of data have to satisfy two conditions:

- $\delta_n\delta_{n-1} = 0 : C_n \to C_{n-2}$ for $n \geqslant 3$.
- $Im(\delta_2)$ acts by conjugation on C_2, and trivially on C_n for $n \geqslant 3$.

Remark 5.6.1.

- C_1 acts on its vertex groups by conjugation.
- For $n \geqslant 3$, since $Im(\delta_2)$ acts trivially on C_n then we have C_1 acts on C_n through

$$\pi_1(C) = \operatorname{coker} \delta_2 = C_1/Im(\delta_2)$$

which is called the *fundamental groupoid* of C.

An ordered morphism of ordered crossed complexes $f : C \to D$

$$\begin{tikzcd}
\ldots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "f_{n-1}"] & \ldots\ldots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[d, "f_1"] \\
\ldots \arrow[r] & D_n \arrow[r, "\gamma_n"] & D_{n-1} \arrow[r, "\gamma_{n-1}"] & \ldots\ldots \arrow[r, "\gamma_3"] & D_2 \arrow[r, "\gamma_2"] & D_1
\end{tikzcd}$$

is a family of ordered functors $f_n : C_n \to D_n$ for $n \geqslant 1$ all inducing the same ordered function of vertices $f_0 : \mathrm{Ob}(C) \to \mathrm{Ob}(D)$, and satisfying two conditions:

1. $cf_n\gamma_n = c\delta_n f_{n-1}$.

2. $(c \triangleleft c_1)f_n = cf_n \triangleleft c_1 f_1$ for all $c \in C_n$ and $c_1 \in C_1$.

We denote by $\mathcal{OCRS}$ the resulting category of ordered crossed complexes.

The category of ordered chain complexes

Let G be an ordered groupoid. A chain complex A over G is a sequence

$$\ldots \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \ldots\ldots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

of G-modules and ordered G-morphisms satisfying $\partial\partial = 0$.

Let $\theta : G \to H$ be an ordered functor of groupoids, and let A, B be ordered chain complexes over G, H respectively. An ordered morphism of chain complexes over θ is $f : A \to B$

$$\begin{tikzcd}
\ldots \arrow[r, "\partial_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & \ldots\ldots \arrow[r, "\partial_2"] & A_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_0"] \\
\ldots \arrow[r, "\eta_{n+1}"] & B_n \arrow[r, "\eta_n"] & B_{n-1} \arrow[r, "\eta_{n-1}"] & \ldots\ldots \arrow[r, "\eta_2"] & B_1 \arrow[r, "\eta_1"] & B_0
\end{tikzcd}$$

consisting of a family of morphisms $f_n : A_n \to B_n$ for $n \geqslant 0$ satisfying that $f_n\eta_n = \partial_n f_{n-1}$. These data form a category $\mathcal{OCHN}$ of chain complexes over ordered groupoids.

The ordered derived chain complex functor

Before we construct the functor $\Delta : \mathcal{OCRS} \to \mathcal{OCHN}$, we need the following definition of abelianisation.

Definition The abelianisation G^{ab} of an ordered groupoid G is the quotient ordered groupoid $G/[G, G]$ where $[G, G]$ is the normal ordered subgroupoid generated by the commutators from all vertex groups of G. If the vertex groups of a groupoid are all abelian groups, we then call it an abelian groupoid.

Remark 5.6.2. $[G, G]$ is a normal ordered subgroupoid because of its structure as a disjoint union of groups, and it satisfies Matthew's definition of normal subgroupoid [26]. Also $Im(\delta_2)$ in the following disscusion is a union of groups, so Matthew's definition works for $Im(\delta_2)$ as well.

We aim now to construct the functor $\Delta : \mathcal{OCRS} \to \mathcal{OCHN}$, and for that purpose let $G = \operatorname{coker}\delta_2 = C_1 / Im(\delta_2)$. That makes C_n, for $n \geqslant 3$, into G-modules, then we can consider ∂_n , from the chain complex, to be just δ_n, from the crossed complex, since C_n are G-modules and δ_n are G-morphisms , but we are then left with mapping

$$C_2 \xrightarrow{\delta_2} C_1$$

to

$$A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

Consider the sequence of ordered groupoids and ordered morphisms

$$C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\varpi} G$$

$\mathrm{Ob}(G)$ is an ordered discrete groupoid. Define $\zeta : C_2 \to \mathrm{Ob}(G)$ to be the ordered trivial functor. An ordered ζ-derivation is a morphism $C_2 \to B$ for some $\mathrm{Ob}(G)$-module B. Hence, B is an abelian ordered groupoid with $B_0 = \mathrm{Ob}(G)$ and a ζ-derivation induces a morphism $C_2^{ab} \to B$. Further, we have the ordered abelianisation map $\alpha_2 : C_2 \to C_2^{ab}$, it follows that $D_\zeta = C_2^{ab}$.

$$\begin{tikzcd} C_2 \arrow[r] \arrow[d, "\alpha_2"'] & B \\ C_2^{ab} \arrow[ur] & \end{tikzcd}$$

Now, if we define $\alpha_1 : C_1 \to D_\varpi$ to be the universal ordered derivation over ϖ to the ordered derived module D_ϖ, and $\alpha_0 : G \to \overrightarrow{\mathbb{Z}}G$ to be the universal ordered derivation $G \to \overrightarrow{I}G$ followed by the inclusion $\overrightarrow{I}G \to \overrightarrow{\mathbb{Z}}G$ then we have the following diagram

$$\begin{tikzcd} C_2 \arrow[r, "\delta_2"] \arrow[d, "\alpha_2"] & C_1 \arrow[r, "\varpi"] \arrow[d, "\alpha_1"] & G \arrow[d, "\alpha_0"] \\ C_2^{ab} & D_\varpi & \overrightarrow{\mathbb{Z}}G \end{tikzcd}$$

We aim now to define $\partial_2 : C_2^{ab} \to D_\varpi$ and $\partial_1 : D_\varpi \to \overrightarrow{\mathbb{Z}}G$. Apply the functor $\mathcal{D} : \mathcal{OG}^2 \to \mathcal{OMOD}$ to the square

$$\begin{tikzcd} C_2 \arrow[r, "\delta_2"] \arrow[d, "\zeta"'] & C_1 \arrow[d, "\varpi"] \\ \mathrm{Ob}(G) \arrow[r] & G \end{tikzcd}$$

to map ζ to D_ζ, ϖ to D_ϖ, and the commutative square to the ordered morphism $D_\zeta \to D_\varpi$ which is $\partial_2 : C_2^{ab} \to D_\varpi$. We need this morphism to be a G-module morphism, so let

$x \in C_2^{ab}$, $g \in G$ and $x \triangleleft g$ exists such that $x = c_2\alpha_2$ and $g = c_1\varpi$, then we have

$$\begin{aligned}(x \triangleleft g)\partial_2 &= (c_2 \triangleleft c_1)\delta_2\alpha_1 \\ &= (c_1^{-1}c_2\delta_2 c_1)\alpha_1 \\ &= [(c_1^{-1}\alpha_1) \triangleleft c_2\delta_2\varpi + (c_2\delta_2)\alpha_1] \triangleleft c_1\varpi + c_1\alpha_1 \\ &= (c_1^{-1}\alpha_1) \triangleleft c_1\varpi + (c_2\delta_2\alpha_1) \triangleleft c_1\varpi + c_1\alpha_1 \\ &= -c_1\alpha_1 + (c_2\delta_2\alpha_1) \triangleleft c_1\varpi + c_1\alpha_1 \\ &= (c_2\alpha_2\partial_2) \triangleleft c_1\varpi \\ &= (x\partial_2) \triangleleft g\end{aligned}$$

For $\partial_1 : D_\varpi \to \overrightarrow{\mathbb{Z}}G$, since we have $\varpi : C_1 \to G$ and $\psi : G \to \overrightarrow{I}G$ then the composition $\varpi\psi$ actually is a ϖ-derivation $C_1 \to \overrightarrow{I}G$. Using the universal property of D_ϖ we find a G-morphism $D_\varpi \to \overrightarrow{I}G$ and ∂_1 is just this G-morphism composed with the inclusion $\overrightarrow{I}G \to \overrightarrow{\mathbb{Z}}G$.

Finally, the relations $\partial_3\partial_2 = \partial_2\partial_1 = 0$ are required for constructing the functor Δ . We have $\alpha_2\partial_2\partial_1 = \delta_2\varpi\alpha_0 = 0$, but α_2 is the abelianisation map so it is surjective which implies that $\partial_2\partial_1 = 0$. Also $\partial_3\partial_2 = \delta_3\delta_2\alpha_1$ and we know that α_1 is a ϖ-derivation and $Im(\delta_2)$ is just $\ker\varpi$, so $\partial_3\partial_2 = 0$.

Now, we can construct the functor $\Delta : \mathcal{OCRS} \to \mathcal{OCHN}$, and for that let C be an ordered crossed complex, and let $\varpi : C_1 \to G$ be cokernel δ_2. Then there are G-morphisms

$$C_2^{ab} \xrightarrow{\partial_2} D_\varpi \xrightarrow{\partial_1} \overrightarrow{\mathbb{Z}}G$$

that make the following diagram commute

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "1"'] & C_{n-1} \arrow[r] \arrow[d, "1"'] & \cdots \arrow[r] & C_3 \arrow[r, "\delta_3"] \arrow[d, "1"'] & C_2 \arrow[r, "\delta_2"] \arrow[d, "\alpha_2"] & C_1 \arrow[r, "\varpi"] \arrow[d, "\alpha_1"] & G \arrow[d, "\alpha_0"] \\
\cdots \arrow[r] & C_n \arrow[r, "\partial_n"'] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_3 \arrow[r, "\partial_3"'] & C_2^{ab} \arrow[r, "\partial_2"'] & D_\varpi \arrow[r, "\partial_1"'] & \overrightarrow{\mathbb{Z}}G
\end{tikzcd}$$

such that the upper line is the ordered crossed complex C and the lower line is $C\Delta$ an ordered chain complex over G.

5.7 The right adjoint of the ordered derived functor

The main task of this section is to construct a functor $\Theta : \mathcal{OCHN} \to \mathcal{OCRS}$ and prove it is a right adjoint of Δ. For that purpose we start with defining the pullback groupoid $P(M, G)$ for an ordered groupoid G and a G-module M, then this construction gives the functor $P : \mathcal{OMOD} \to \mathcal{OG}$ wich is a right adjoint of the functor $\overrightarrow{\mathbb{Z}}(-)$. After that we prove that the ordered natural transformation $\kappa : P(M, G) \to G \ltimes M$ is an ordered covering morphism of groupoids. Then we define the ordered crossed complex $A\Theta'$ over the ordered chain complex A. Finally, we construct the functor Θ as the pullback of the crossed complex $A\Theta'$ along κ and prove that it is a right adjoint of Δ.

Definition Given an ordered groupoid G with M as a G-module and consider M as a poset ΩM with the ordered target map $t : \Omega M \to Ob(G)$. Then the pullback groupoid $P(M, G)$ is the set:

$$P(M, G) = \{(m, g, n) : mt = d(g), nt = r(g)\}$$

$$\begin{tikzcd}
P(M,G) \arrow[r] \arrow[d] & G \boxtimes \Omega M \arrow[r] \arrow[d] & \Omega M \arrow[d, "t"] \\
\Omega M \boxtimes G \arrow[r] \arrow[d] & G \arrow[r, "r"'] \arrow[d, "d"] & \mathrm{Ob}(G) \\
\Omega M \arrow[r, "t"'] & \mathrm{Ob}(G) &
\end{tikzcd}$$

with groupoid structure defined as follows:

- $d(m, g, n) = mt, r(m, g, n) = nt.$
- $(m, g, n)(n, h, k) = (m, gh, k).$
- $(m, g, n)^{-1} = (n, g^{-1}, m), 1_m = (m, 1_{mt}, m).$

This construction gives a functor $P : \mathcal{OMOD} \to \mathcal{OG}$.

Remark 5.7.1. $P(M, G)$ is an ordered groupoid since it is the pullback of an ordered groupoid G over an ordered function t.

Lemma 5.7.2. *The functor $P : \mathcal{OMOD} \to \mathcal{OG}$ is a right adjoint of $\overrightarrow{\mathbb{Z}}(-) : \mathcal{OG} \to \mathcal{OMOD}$.*

Proof. We wish to establish the adjunction between

$$P : \mathcal{OMOD} \to \mathcal{OG}$$

and

$$\overrightarrow{\mathbb{Z}} : \mathcal{OG} \to \mathcal{OMOD}.$$

Let M be a G–module, with $w : M \to \mathrm{Ob}(G)$. For an ordered groupoid H, a functor ϕ from H to $P(M,G)$ is determined by an ordered mapping $\alpha : H_0 \to \Omega M$ and an ordered functor $\theta : H \to G$ such that, for all $x \in H_0$, we have $\theta = \alpha w$. Then

$$h\phi = (d(h)\alpha, h\theta, r(h)\alpha).$$

From this data, we define a functor $\gamma : \overrightarrow{\mathbb{Z}}H \to M$ as follows. It suffices to define γ on the arrows of H, and we set

$$h\gamma = d(h)\alpha \lhd h\theta\,,$$

where $\lhd$ is the G–action on M. We note that $h\gamma \in M_{r(h\theta)} = M_{r(h)\theta}$, and γ is clearly ordered. Moreover, for composable $h, k \in H$ we have

$$(h \lhd k)\gamma = (hk)\gamma = (d(h)\alpha) \lhd (hk)\theta = (d(h)\alpha) \lhd h\theta \lhd k\theta = h\gamma \lhd k\theta\,,$$

and so γ is a functor in $\mathcal{OMOD}(\overrightarrow{\mathbb{Z}}H, M)$.

For the inverse, suppose we are given a functor $\overrightarrow{\mathbb{Z}}H \to M$ in $\mathcal{OMOD}$: so we have $\theta : H \to G$ and $\mu : \overrightarrow{\mathbb{Z}}H \to M$ such that, for any composable $h, k \in H$ we have

$$(h \lhd k)\mu = (hk)\mu = h\mu \lhd k\theta\,.$$

To define the corresponding $\eta : H \to P(M,G)$, we use θ as given and μ restricted to H_0, and define

$$h\eta = (d(h)\mu, h\theta, r(h)\mu)\,.$$

If we use η as just defined by μ to determine $\gamma : \overrightarrow{\mathbb{Z}}H \to M$, then

$$\gamma : h \mapsto d(h)\mu \lhd h\theta = (d(h) \lhd h)\mu = h\mu$$

and so $\gamma = \mu$.

On the other hand, if we begin with ϕ and determine $\gamma : h \mapsto d(h)\alpha \lhd h\theta$, then this γ determines $\eta : h \mapsto (d(h)\gamma, h\theta, r(h)\gamma)$, where for any $x \in H_0$ we have $x\gamma = x\alpha \lhd x\theta = x\alpha \lhd x\alpha w = x\alpha$, and so

$$h\eta = (d(h)\alpha, h\theta, r(h)\alpha) = h\phi$$

and so $\eta = \phi$. $\square$

Proposition 5.7.3. *Let $\theta : H \to G$ be an ordered morphism of groupoids, and M a G-*

module. Then for a commutative triangle in the category $\mathcal{OMOD}$

$$\begin{tikzcd} \overrightarrow{I}H \arrow[r, "i"] \arrow[d, "\tilde{f}"'] & \overrightarrow{\mathbb{Z}}H \arrow[dl, "\gamma"] \\ M & \end{tikzcd}$$

we have $F \in \mathcal{OG}(H, G \ltimes M)$ from the adjunction of Proposition 5.3.3, $\eta \in \mathcal{OG}(H, P(M,G))$ from the adjunction of Lemma 5.7.2 and there exists κ that makes the following triangle commute in the category $\mathcal{OG}$

$$\begin{tikzcd} G \ltimes M & P(M,G) \arrow[l, "\kappa"'] \\ H \arrow[u, "F"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

such that $\kappa : P(M,G) \to G \ltimes M$ given by $(m,g,n)\kappa = (g, (m \triangleleft g) - n)$, and for any module Mover an ordered groupoid G, this κ is an ordered covering morphism of groupoids.

Proof. If $h \in H(p,q)$, then

$$hF = (h\theta, h\psi\tilde{f}) = (h\theta, (h - 1_q)\tilde{f}) \text{ and } h\eta = (1_p\gamma, h\theta, 1_q\gamma)$$

Now, we may take $G = H$, $\theta = 1_G$, and $g \in G(p,q)$. Consider $g\eta = (1_p\gamma, g, 1_q\gamma) = (m, g, n) \in P(M,G)$. Then

$$\begin{aligned} (m,g,n)\kappa &= gF \\ &= (g\theta, (g - 1_q)\tilde{f}) \\ &= (g, (g - 1_q)\gamma) \\ &= (g, (1_p g)\gamma - 1_q\gamma) \\ &= (g, (m \triangleleft g) - n). \end{aligned}$$

Finally, let $(g,x) \in G \ltimes M$ with $g \in G(p,q)$ and $x \in M_q$, and let $m \in M_p$ be an object of $P(M,G)$ lying over the source p of (g,x). Then there is a unique $n \in M_q$ such that $(m \triangleleft g) - n = x$. Hence there is a unique element (m,g,n) over (g,x). □

Definition For an ordered chain complex A over an ordered groupoid H, $A\Theta'$ is the ordered crossed complex

$$\ldots \longrightarrow A_n \xrightarrow{\partial_n} A_{n-1} \longrightarrow \ldots \longrightarrow A_3 \xrightarrow{\partial_3} A_2 \xrightarrow{(0,\partial_2)} H \ltimes A_1.$$

Where $H \ltimes A_1$ acts on A_n for $n \geqslant 2$ via the projection $H \ltimes A_1 \to H$, so that A_1 acts trivially.

Note that A_0 does not appear in $A\Theta'$, so we bring A_0 in another construction $A\Theta$ defined as follows .

Definition For any ordered chain complex A, we consider the ordered canonical covering morphism

$$\kappa : P(A_0, H) \to H \ltimes A_0$$

We define

$$A\Theta = A\Theta' \boxtimes \kappa \, ,$$

the pull back along κ of the crossed complex $A\Theta'$.
We obtain a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & (A_3)\Theta \arrow[r] \arrow[d, "\sigma_3"] & (A_2)\Theta \arrow[r] \arrow[d, "\sigma_2"] & (A_1)\Theta \arrow[r] \arrow[d, "\sigma_1"] & P(A_0, H) \arrow[d, "\kappa"] \\
\ldots \arrow[r] & A_3 \arrow[r, "\partial_3"'] & A_2 \arrow[r, "(0,\partial_2)"'] & H \ltimes A_1 \arrow[r, "(1,\partial_1)"'] & H \ltimes A_0
\end{tikzcd}$$

in which each $(A_n)\Theta$ is an ordered groupoid over $(A_0)\Theta = A_0$, and each σ_n is an ordered covering morphism of groupoids.

The lower sequence in the diagram above is constructed over the chain complex A, so the composition $A_n \to H \ltimes A_0$ is 0 for $n \geqslant 2$. In addition, since σ_n is an ordered covering morphism it has a discrete kernel so we have $(A_n)\Theta$ as a family of groups each isomorphic to a group of A_n for $n \geqslant 2$. There is also an action of $(A_1)\Theta$ on $(A_n)\Theta$ for $n \geqslant 2$ induced by the action of $H \ltimes A_1$ on A_n ; for if $e_1 \in (A_1)\Theta(x, y)$, where $x \in (A_0)_p$, $y \in (A_0)_q$, and if $e_n \in ((A_n)\Theta)_x$, then $e_1\sigma_1$ acts on $e_n\sigma_n$ to give an element of $(A_n)_q$ which lifts uniquely to an element of $((A_n)\Theta)_y$.
It is now clear that $(A)\Theta = \{(A_n)\Theta\}_{n\geqslant 0}$ is an ordered crossed complex and that the σ_i form an ordered morphism $\sigma : A\Theta \to A\Theta'$ of ordered crossed complexes. This gives a functor

$$\Theta : \mathcal{OCHN} \to \mathcal{OCRS}.$$

Using the constructions given above, we can extract an explicit description of $A\Theta$ as follows:
Recall that A_0 is an H-module and so comes with an ordered function $A_0 \to H_0$ making A_0 the disjoint union of abelian groups $(A_0)_p$. The set of objects of every $(A_n)\Theta$ is just A_0, regarded as a poset and as a disjoint union.
An arrow of $(A_1)\Theta$ from x to y, where $x \in (A_0)_p$, $y \in (A_0)_q$, $p, q \in H_0$, is a triple (h, a, y), where $h \in H(p, q)$, $a \in (A_1)_q$, and $x \triangleleft h = y + a\partial_1$. Composition in $(A_1)\Theta$ is given by

$$(k, b, x)(h, a, y) = (kh, b \triangleleft h + a, y)$$

for $(k, b, x) \in (A_1\Theta)(z, x)$ and $(h, a, y) \in (A_1\Theta)(x, y)$.
The ordering is defined as $(h, a, y) \leqslant (k, b, z)$ if and only if $h \leqslant k$ in H.
For $n \geqslant 2$, $(A_n)\Theta$ is a family of groups; the group at the object $x \in (A_0)_p$ has elements (b, x) where $b \in (A_n)_p$, with composition

$$(b, x) + (c, x) = (b + c, x).$$

with ordering defined as $(c, y) \leqslant (b, x)$ if and only if $q \leqslant p$ where $x \in (A_0)_p, y \in (A_0)_q$.
The boundary map $\delta_2 : (A_2)\Theta \to (A_1)\Theta$ is clearly ordered and given by

$$(b, x)\delta = (1_p, b\partial_2, x) \text{ for } b \in (A_2)_p \text{ and } x \in (A_0)_p.$$

The boundary map $\delta_n : (A_n)\Theta \to (A_{n-1})\Theta$ for $n \geqslant 3$ is also ordered and given by $(b, x)\delta = (b\partial_n, x)$ and the action of $(A_1)\Theta$ on $(A_n)\Theta$ with $n \geqslant 2$ is given by

$$(b, x) \triangleleft (h, a, y) = (b \triangleleft h, y)$$

where $h \in H(p, q)$, $a \in (A_1)_q$, $y \in (A_0)_q$ and $x \triangleleft h = y + a\partial$.

Proposition 5.7.4. *The functor Θ is a right adjoint of Δ.*

Proof. Let C be an ordered crossed complex with derived ordered chain complex $C\Delta$ over the ordered groupoid G, A an ordered chain complex over the ordered groupoid H, and $\psi : G \to H$ an ordered morphism of groupoids. An ordered morphism $\beta : C\Delta \to A$ in $\mathcal{OCHN}$ is equivalent to a commutative diagram in $\mathcal{OMOD}$

$$\begin{tikzcd}
\dots \arrow[r] & C_3 \arrow[r] \arrow[d, "\beta_3"] & C_2^{ab} \arrow[r] \arrow[d, "\beta_2"] & D_\varpi \arrow[r] \arrow[d, "\beta_1"] & \overrightarrow{I}G \arrow[r] \arrow[d, "\acute{\beta_0}"] & \overrightarrow{\mathbb{Z}}G \arrow[dl, "\beta_0"] \\
\dots \arrow[r] & A_3 \arrow[r, "\partial_3"'] & A_2 \arrow[r, "\partial_2"'] & A_1 \arrow[r, "\partial_1"'] & A_0 &
\end{tikzcd}$$

And by propositions 5.3.3 and Lemma 5.7.2 to a commutative diagram in $\mathcal{OG}$

$$\begin{tikzcd}
\dots \arrow[r] & C_3 \arrow[r] \arrow[d, "\beta_3"] & C_2 \arrow[r] \arrow[d, "\bar{\beta}_2"] & C_1 \arrow[r, "\varpi"] \arrow[d, "\gamma_1"] & G \arrow[d, "\xi"'] \arrow[dr] & \\
\dots \arrow[r] & A_3 \arrow[r, "\partial_3"'] & A_2 \arrow[r, "(0, \partial_2)"'] & H \ltimes A_1 \arrow[r, "(1, \partial_1)"'] & H \ltimes A_0 & P(A_0, H) \arrow[l, "\kappa"']
\end{tikzcd}$$

where $(..., \beta_3, \bar{\beta}_2, \gamma_1)$ is an ordered morphism of ordered crossed complexes, and κ is the canonical ordered covering morphism. This in turn is equivalent to a commutative diagram

$$\begin{tikzcd}
... \arrow[r] & C_3 \arrow[r] \arrow[d, "\beta_3"] & C_2 \arrow[r, "\delta"] \arrow[d, "\bar{\beta}_2"] & C_1 \arrow[r, "\omega"] \arrow[d, "\gamma_1"] & P(A_0, H) \arrow[d, "\kappa"] \\
... \arrow[r] & A_3 \arrow[r] & A_2 \arrow[r] & H \ltimes A_1 \arrow[r] & H \ltimes A_0
\end{tikzcd}$$

because, in any such diagram, $\delta\omega\kappa = 0$ and κ is a covering morphism, so $\delta\omega = 0$, that is, ω factorises through ϖ.

This diagram is therefore equivalent to an ordered morphism of crossed complexes $C \to A\Theta$. Hence β is equivalent to an ordered morphism of crossed complexes $C \to A\Theta$. This shows that Θ is right adjoint to Δ. $\square$

Chapter 6

Cohomology of crossed modules over groups

This chapter consists of three main sections, the first one gives the basic concepts and relations between a crossed module $\mu : A \to G$, a derivation $d : G \to A$ and the monoid $End(A, G, \mu)$. In addition, it describes the monoid of all derivations $Der(G, A)$ over the crossed module μ with some usefull properties satisfied in this monoid where the inner derivations χ_a play an important role. These ideas are all taken from the *algebraic digression* in [32, section 7].

In the second section, we use the monoid $Der(G, A)$ as the vertex set of the groupoid $Z^1(G, A)$ and identify the set of arrows using the properties proved in the first section. The groupoid $Z^1(G, A)$ is introduced in [3, section 5] for any action of a group G on a group A: in the crossed module case we obtain extra structure. Brown [3] defines the 1–dimensional cohomolgy set $H^1(G, A)$ as the set $\pi_0(Z^1(G, A))$: we obtain a monoid structure on $H^1(G, A)$ from the monoid structure of $Der(G, A)$. We derive an exact sequences in cohomology from a short exact sequence of crossed modules, getting an exact sequence of monoids rather than of cohomology sets as in [3], and generalising that given in [14] which only considered the groups of units. Moreover, we characterise the units in $Der(G, A)$ and link the group $L^1(G, A)$ from [24] to the group of units in $H^1(G, A)$.

The third section is all about derivations linked to endomorphisms of the crossed module μ, we construct the monoid $Der_\diamond(G, A)$ of all such derivations and use it as the arrow set to construct the groupoid $Gpd_\diamond(\mu, \mu)$ where the vertex set is just $End(A, G, \mu)$. Furthermore, we verify the Eckmann-Hilton condition to prove that $Gpd_\diamond(\mu, \mu)$ is a monoidal groupoid. The results of this section will be generalised to ordered crossed modules of groupoids in chapter 7, using the monoidal closed structure on the category of crossed complexes set out in [9].

6.1 Crossed modules, derivations and monoids

This section provides the main concepts and constructions needed in later sections. Starting with a group crossed module $\mu : A \to G$ and a derivation $d : G \to A$, we define an endomorphism $\langle \alpha_d, \theta_d \rangle$ of the crossed module μ. Then we construct the monoid of all derivations $Der(G, A)$ over the crossed module μ. After that we define the inner derivation χ_a and derive the composition $d * \chi_a$ in $Der(G, A)$.

Definition Recall that a *crossed module* (of groups) (A, G, μ) is a morphism $\mu : A \to G$ of groups together with an action of the group G on the right of the group A, written $(a, g) \mapsto a \triangleleft g$, satisfying the rules:

CM1 $(a \triangleleft g)\mu = g^{-1}(a\mu)g$,

CM2 $b \triangleleft a\mu = a^{-1}ba$,

for all $g \in G$, $a, b \in A$.

Example 6.1.1.
- Suppose N is a normal subgroup of the group G. Then G acts on N by conjugation, this action and the inclusion map $i : N \hookrightarrow G$ form a crossed module (N, G, i).
  Since any group G has the two normal subgroups $\{1\}$ and G, we can form the conjugation crossed modules $\{1\} \hookrightarrow G$ and $id : G \to G$ and so any group G may be thought of as a crossed module in two ways.
- If M is a G-module, then the action of G on M together with the zero homomorphism $0 : M \to G$, sending all elements of M to the identity in G, yields a G-crossed module $(M, G, 0)$.
- Let G be any group $Aut(G)$ its group of automorphisms. There is an obvious action of $Aut(G)$ on G, and a homomorphism $f : G \to Aut(G)$ sending each $g \in G$ to the inner automorphism of conjugation by g. These together form the crossed module $(G, Aut(G), f)$.

The axioms of the crossed module impose some general properties on $\ker \mu$ and $Im(\mu)$ which are given in the following proposition.

Proposition 6.1.2. *Let $\mu : A \to G$ be a crossed module. Then*

(i) $A\mu$ is a normal subgroup of G.

(ii) $\ker \mu$ is central in A.

(iii) $A\mu$ acts trivially on the center $Z(A)$ of A.

(iv) $Z(A)$ *and* $\ker \mu$ *are* $\operatorname{coker} \mu$*-modules.*

Proof. (i) $g^{-1}(a\mu)g = (a \triangleleft g)\mu \in A\mu$.

(ii) Let $a, b \in A$ with $a\mu = 1_G$, then $b \triangleleft a\mu = a^{-1}ba = b$ so $ba = ab$.

(iii) If $b \in Z(A)$ and $a \in A$, then $b \triangleleft a\mu = b$.

(iv) Follows using $(i), (ii)$ and (iii) with considering $Cok\mu = G/A\mu$.

□

Definition If $\mu_1 : A_1 \to G_1$ and $\mu_2 : A_2 \to G_2$ are crossed modules, then a crossed module morphism $\Gamma : \mu_1 \to \mu_2$ consists of a pair $\langle \Gamma_A, \Gamma_G \rangle$ of group homomorphisms $\Gamma_A : A_1 \to A_2$ and $\Gamma_G : G_1 \to G_2$ that commute with μ_i and preserve the action. That is

$$\begin{array}{ccc} A_1 & \xrightarrow{\mu_1} & G_1 \\ {\scriptstyle \Gamma_A}\downarrow & & \downarrow{\scriptstyle \Gamma_G} \\ A_2 & \xrightarrow[\mu_2]{} & G_2 \end{array}$$

$\Gamma_A\mu_2 = \mu_1\Gamma_G$ and $(a_1 \triangleleft g_1)\Gamma_A = a_1\Gamma_A \triangleleft g_1\Gamma_G$.

Remark 6.1.3. If $\mu_1 = \mu_2 = \mu$ then Γ is an endomorphism of the crossed module μ, and Γ_A, Γ_G are endomorphisms of A and G respectively, so we have $\Gamma_A\mu = \mu\Gamma_G$ and $(a \triangleleft g)\Gamma_A = a\Gamma_A \triangleleft g\Gamma_G$. The set of such endomorphisms form the monoid $\mathrm{End}(A, G, \mu)$.

Definition A mapping $d : G \to A$ is a *derivation* if for all g, h in G

$$(gh)d = (gd \triangleleft h)hd.$$

Let $\mu : A \to G$ be a crossed module, and $d : G \to A$ a derivation, then d determines the mapping $\theta_d : G \to G$ defined as $g \mapsto g(gd\mu)$. This mapping is an endomorphism of G, $\theta_d \in \mathrm{End}(G)$, since if $g, h \in G$ then we have

$$\begin{aligned} (gh)\theta_d &= gh(gh)d\mu \\ &= gh[(gd \triangleleft h)hd]\mu \\ &= gh(gd \triangleleft h)\mu(hd\mu) \\ &= ghh^{-1}(gd\mu)h(hd\mu) \\ &= g(gd\mu)h(hd\mu) \\ &= (g\theta_d)(h\theta_d). \end{aligned}$$

d also determines the endomorphism $\alpha_d : A \to A$ given by $a \mapsto a(a\mu d)$, as an element of the monoid $\mathrm{End}(A)$, since for all $a, b \in A$

$$\begin{aligned}
(ab)\alpha_d &= ab(ab)\mu d \\
&= ab(a\mu b\mu)d \\
&= ab(a\mu d \triangleleft b\mu)b\mu d \\
&= abb^{-1}(a\mu d)b(b\mu d) \\
&= a(a\mu d)b(b\mu d) \\
&= (a\alpha_d)(b\alpha_d).
\end{aligned}$$

Lemma 6.1.4. *The pair $\langle \alpha_d, \theta_d \rangle$ is an endomorphism of the crossed module μ.*

Proof. We first prove the commutativity of the square

$$\begin{tikzcd}
A \arrow[r, "\mu"] \arrow[d, "\alpha_d"'] & G \arrow[d, "\theta_d"] \\
A \arrow[r, "\mu"'] & G
\end{tikzcd}$$

$$\begin{aligned}
a\alpha_d\mu &= (a(a\mu d))\mu \\
&= (a\mu)(a\mu d\mu) \\
&= a\mu\theta_d
\end{aligned}$$

then we have

$$\begin{aligned}
(a \triangleleft g)\alpha_d &= (a \triangleleft g)(a \triangleleft g)\mu d \\
&= (a \triangleleft g)(g^{-1}a\mu g)d \\
&= (a \triangleleft g)[(g^{-1}a\mu)d \triangleleft g]gd \\
&= (a \triangleleft g)[(g^{-1}d \triangleleft a\mu)a\mu d \triangleleft g]gd \\
&= (a \triangleleft g)[(a^{-1}g^{-1}da)a\mu d \triangleleft g]gd \\
&= (gd)^{-1}[a(a\mu d) \triangleleft g]gd \\
&= a(a\mu d) \triangleleft g \triangleleft gd\mu \\
&= a(a\mu d) \triangleleft g(gd\mu) \\
&= a\alpha_d \triangleleft g\theta_d.
\end{aligned}$$

so we have $\langle \alpha_d, \theta_d \rangle \in \mathrm{End}(A, G, \mu)$. $\square$

The monoid $Der(G, A)$

The set of all derivations $G \to A$ is denoted by $Der(G, A)$. Elements of $Der(G, A)$ are composable by using the operation $*$, given by

$$g(d * f) = (gd)(g\theta_d f)$$

for each $g \in G$ and $d, f \in Der(G, A)$.

Lemma 6.1.5. *An alternative definition of the composition $*$ is given by*

$$(gd)(g\theta_d f) = (gf)(gd\alpha_f)$$

Proof.

$$\begin{aligned}(gd)(g\theta_d f) &= (gd)[g(gd\mu)]f \\ &= (gd)[gf \triangleleft gd\mu]gd\mu f \\ &= (gd)(gd)^{-1}(gf)(gd)(gd\mu f) \\ &= (gf)(gd\alpha_f).\end{aligned}$$

□

Lemma 6.1.6. *The set $Der(G, A)$ is a monoid under the binary operation $*$.*

Proof. We have $(d * f) \in Der(G, A)$ for all $d, f \in Der(G, A)$ since

$$\begin{aligned}(gh)(d * f) &= ((gh)d)((gh)\theta_d f) \\ &= [(gd \triangleleft h)hd][(g\theta_d f \triangleleft h\theta_d)h\theta_d f] \\ &= [(gd \triangleleft h)hd][(g\theta_d f \triangleleft (h)(hd\mu))h\theta_d f] \\ &= (gd \triangleleft h)(hd)(hd)^{-1}(g\theta_d f \triangleleft h)(hd)(h\theta_d f) \\ &= [(gd)(g\theta_d f) \triangleleft h](hd)(h\theta_d f) \\ &= [(g)d * f \triangleleft h][(h)d * f]\end{aligned}$$

$*$ is associative since we have

$$g((d * f) * k) = ((g)d * f)(g\theta_{d*f}k) = (gd)(g\theta_d f)(g\theta_{d*f}k)$$

and

$$g(d * (f * k)) = (gd)(g\theta_d(f * k)) = (gd)(g\theta_d f)(g\theta_d\theta_f k)$$

But

$$
\begin{aligned}
g\theta_d\theta_f &= (g\theta_d)\theta_f \\
&= (g\theta_d)(g\theta_d f\mu) \\
&= g(gd\mu)[g(gd\mu)]f\mu \\
&= g(gd\mu)[(gf \triangleleft gd\mu)(gd\mu f)]\mu \\
&= g(gd\mu)(gf \triangleleft gd\mu)\mu(gd\mu f)\mu \\
&= g(gd\mu)(gd\mu)^{-1}gf\mu(gd\mu)(gd\mu f)\mu \\
&= g(gf\mu)(gd\mu)(gd\mu f)\mu \\
&= g[(gf)(gd)(gd\mu f)]\mu \\
&= g[(gd)(gd)^{-1}(gf)(gd)(gd\mu f)]\mu \\
&= g[(gd)(gf \triangleleft gd\mu)(gd\mu f)]\mu \\
&= g[(gd)[g(gd\mu)]f]\mu \\
&= g[(gd)(g\theta_d f)]\mu \\
&= g[g(d * f)]\mu \\
&= g\theta_{d*f}
\end{aligned}
$$

The identity element of $Der(G, A)$ is the trivial derivation $\tau : g \mapsto e_A$, where e_A is the identity element in A. $\square$

Remark 6.1.7. The maps $\theta : d \mapsto \theta_d$ and $\alpha : d \mapsto \alpha_d$ are monoid morphisms from the monoid of all derivations $Der(G, A)$ to the endomorphism monoids $End(G)$ and $End(A)$ respectively. So, the map $Der(G, A) \to End(A, G, \mu)$ mapping $d \mapsto \langle \alpha_d, \theta_d \rangle$ is a monoid morphism.

Lemma 6.1.8. *A morphism of crossed G-modules $u : A \to B$ induces a monoid morphism $U : Der(G, A) \to Der(G, B)$.*

Proof. Define U on the derivation $d \in Der(G, A)$ as the composition $du : G \to A \to B$, we want U to be a monoid morphism, that is for all $d, f \in Der(G, A)$ we want $(d * f)U = dU * fU$. Consider the following commutative square

$$
\begin{tikzcd}
A \arrow[r, "\mu_1"] \arrow[d, "u"'] & G \arrow[d, "1"] \\
B \arrow[r, "\mu_2"'] & G
\end{tikzcd}
$$

$$\begin{aligned}
g((d*f)U) &= ((gd)(g\theta_d f))U \\
&= ((gd)(g(gd\mu_1 f)))U \\
&= (gd)U(g(gd\mu_1)f)U \\
&= g(dU)(g(gdu\mu_2))fU \\
&= g(du)(g(gdu\mu_2)fu \\
&= (gdu)(g\theta_{dU})fu \\
&= g(dU * fU)
\end{aligned}$$

Moreover

$$g(\tau_A U) = (g\tau_A)u = (e_A)u = e_B = g\tau_B.$$

□

Definition The *inner derivation* $\chi_a : G \to A$ is the derivation determined by $a \in A$, defined by $g \mapsto (a^{-1} \triangleleft g)a$.

To show that χ_a is a derivation:

$$\begin{aligned}
(gh)\chi_a &= (a^{-1} \triangleleft gh)a \\
&= ((a^{-1} \triangleleft g) \triangleleft h)a \\
&= (((a^{-1} \triangleleft g)aa^{-1}) \triangleleft h)a \\
&= ((a^{-1} \triangleleft g)a) \triangleleft h(a^{-1} \triangleleft h)a \\
&= (g\chi_a \triangleleft h)(h\chi_a)
\end{aligned}$$

Lemma 6.1.9. *The corresponding morphisms θ_{χ_a} and α_{χ_a} are the conjugation maps by $a\mu$ and a respectively.*

Proof.

$$\begin{aligned}
g\theta_{\chi_a} &= g(g\chi_a\mu) \\
&= g[(a^{-1} \triangleleft g)a]\mu \\
&= g(a^{-1} \triangleleft g)\mu(a\mu) \\
&= gg^{-1}(a^{-1}\mu)g(a\mu) \\
&= a\mu^{-1}ga\mu
\end{aligned}$$

$$\begin{aligned}
x\alpha_{\chi_a} &= x(x\mu\chi_a) \\
&= x(a^{-1} \triangleleft x\mu)a \\
&= xx^{-1}a^{-1}xa \\
&= a^{-1}xa
\end{aligned}$$

□

Lemma 6.1.10. *For $a, b \in A$ the composition of inner derivations $\chi_a * \chi_b$ is the inner derivation χ_{ab}.*

Proof.

$$\begin{aligned}
g(\chi_a * \chi_b) &= (g\chi_a)(g\theta_{\chi_a}\chi_b) \\
&= (a^{-1} \triangleleft g)a[(a\mu)^{-1}g(a\mu)]\chi_b \\
&= (a^{-1} \triangleleft g)a(b^{-1} \triangleleft (a\mu)^{-1}g(a\mu))b \\
&= (a^{-1} \triangleleft g)a(a^{-1}b^{-1} \triangleleft (a\mu)^{-1}g)ab \\
&= (a^{-1} \triangleleft g)a(aa^{-1}b^{-1}a^{-1} \triangleleft g)ab \\
&= (a^{-1} \triangleleft g)(ab^{-1}a^{-1} \triangleleft g)ab \\
&= (a^{-1}ab^{-1}a^{-1} \triangleleft g)ab \\
&= (b^{-1}a^{-1} \triangleleft g)ab \\
&= g\chi_{ab}.
\end{aligned}$$

□

The following two Lemmas detail the composition of a derivation d and an inner derivation χ_a in $Der(G, A)$.

Lemma 6.1.11. *If $d \in Der(G, A)$ is a derivation and $\chi_a \in Der(G, A)$ is an inner derivation determined by $a \in A$ then*

$$g(d * \chi_a) = (a^{-1} \triangleleft g)(gd)a$$

Proof.

$$\begin{aligned}
g(d * \chi_a) &= (g\chi_a)(gd\alpha_{\chi_a}) \\
&= ((a^{-1} \triangleleft g)a)(a^{-1}(gd)a) \\
&= (a^{-1} \triangleleft g)(gd)a.
\end{aligned}$$

□

Lemma 6.1.12. *If $\chi_a \in Der(G, A)$ determined by $a \in A$ and $d \in Der(G, A)$, then*

$$\chi_a * d = d * \chi_{a\alpha_d}.$$

Proof.

$$\begin{aligned}
g(\chi_a * d) &= (g\chi_a)(g\theta_{\chi_a} d) \\
&= (a^{-1} \triangleleft g)a[(a^{-1}\mu)g(a\mu)]d \\
&= (a^{-1} \triangleleft g)a[a^{-1}\mu d \triangleleft g(a\mu)](g(a\mu)d) \\
&= (a^{-1} \triangleleft g)aa^{-1}(a^{-1}\mu d \triangleleft g)a(g(a\mu))d \\
&= (a^{-1} \triangleleft g)(a^{-1}\mu d \triangleleft g)a(g(a\mu))d \\
&= (a^{-1}(a^{-1}\mu d) \triangleleft g)a(g(a\mu))d \\
&= ((a^{-1}\alpha_d) \triangleleft g)a((gd) \triangleleft a\mu)a\mu d \\
&= ((a^{-1}\alpha_d) \triangleleft g)aa^{-1}(gd)a(a\mu d) \\
&= ((a^{-1}\alpha_d) \triangleleft g)(gd)a\alpha_d \\
&= g(d * \chi_{a\alpha_d})
\end{aligned}$$

□

6.2 Cohomology of crossed modules over groups

In this section we construct the groupoid $Z^1(G, A)$ as the fibre over 1_G of the fibration $p_* : \mathsf{GPD(G, G \ltimes A)} \to \mathsf{GPD(G, G)}$ following the approach of [3]. In addition, we consider $Der(G, A)$ as the vertex set of the groupoid $Z^1(G, A)$ and identify the set of arrows $Z^1(d, f)$ using Lemma 6.1.11. We also use Lemma 6.1.12 to link the derivation d to the composition $\chi_a * d$. After that we describe a monoid structure on the cohomology set $H^1(G, A)$ induced by the monoid structure of $Der(G, A)$. As in [3] if $1 \to A \to B \to C \to 1$ is an exact sequence of G-modules, then the fundamental exact sequence of non-abelian cohomology is of the form $1 \to A^G \to B^G \to C^G \to H^1(G, A) \to H^1(G, B) \to H^1(G, C)$ and it is six term sequence of groups and based sets. We verify that this sequence preserves the monoid structure of the last three terms in the crossed module case. At the end of this section, we characterise the group $Der^*(G, A)$ of units in $Der(G, A)$ using the endomorphisms θ_d and α_d for $d \in Der^*(G, A)$, and we prove that the quotient group $L^1(G, A) = Der^*(G, A)/Ider(G, A)$ is just the group of units of $H^1(G, A)$.

The groupoid $Z^1(G, A)$

Let (A, G, μ) be a crossed G-module. We can form the split extension $G \ltimes A$ which is a group with elements of the form (g, a) and multiplication defined as follows:

$$(g, a)(h, b) = (gh, (a \triangleleft h)b)$$

Now, define $i : A \to G \ltimes A$ as $a \mapsto (e_G, a)$ to be the inclusion where e_G is the identity element of the group G, and $p : G \ltimes A \to G$ as $(g, a) \mapsto g$ to be the projection onto G. Obviously i is injective and p is surjective, so we can form the split exact sequence

$$1 \longrightarrow A \xrightarrow{\ i\ } G \ltimes A \xrightarrow{\ p\ } G \longrightarrow 1$$

Let $s : G \to G \ltimes A$ given by $g \mapsto (g, g\bar{s})$ be a section of p, that is $sp = 1_G$. The composition of s with projection onto A is a function $\bar{s} : G \to A$ called the principal part of s. Observe that $\bar{s}$ is not a morphism since $(gh)s = (gh, (a \triangleleft h)b)$ and $(gh)\bar{s} = (a \triangleleft h)b = (g\bar{s} \triangleleft h)h\bar{s}$, so $\bar{s}$ is a derivation.

Example 6.2.1. Consider $s_1 : G \to G \ltimes A$ mapping $g \mapsto (g, e_A)$ as a section of p, the principal part of s_1 is the trivial derivation $\bar{s_1} = \tau : G \to A$ taking $g \mapsto e_A$.

Remark 6.2.2. The set of derivations is nonempty since at least it contains $\bar{s_1}$.

Following [3, section 5], to define the groupoid $Z^1(G, A)$ recall that the internal hom groupoid $\mathsf{GPD(G, G \ltimes A)}$ has set of objects as all group morphisms $G \to G \ltimes A$ and arrows are natural transformations between morphisms. Define $p_* : \mathsf{GPD(G, G \ltimes A)} \to \mathsf{GPD(G, G)}$ as a morphism of groupoids induced by the projection p such that p_* takes a group morphism as an object of the groupoid $\mathsf{GPD(G, G \ltimes A)}$ and composes it with p to obtain an endomorphism of G which is an object in $\mathsf{GPD(G, G)}$. On arrows, a natural transformation determined by (h, b) in $G \ltimes A$ is mapped by p_* to a natural transformation determined by $h \in G$. Then p_* is a fibration [3, Proposition 2.9], the identity $1_G : G \to G$ is an object of $\mathsf{GPD(G, G)}$, and the fibre of p_* over 1_G is written as $Z^1(G, A)$. Clearly, $Z^1(G, A)$ is a subgroupoid of $\mathsf{GPD(G, G \ltimes A)}$.

Let $s, t \in Ob(Z^1(G, A))$ be two sections of p, then $Z^1(s, t)$ is the set of natural transformations between these two objects which are mapped by p_* to the identity $1_G : G \to G$. That is, $Z^1(s, t)$ is a set of elements $(e_G, a) \in G \ltimes A$ that satisfy the following condition:

$$\begin{aligned}
gt &= (e_G, a)^{-1} gs (e_G, a) \\
(g, g\bar{t}) &= (e_G, a)^{-1} (g, g\bar{s})(e_G, a) \\
(g, g\bar{t}) &= (g, (a^{-1} \triangleleft g)(g\bar{s}))(e_G, a) \\
(g, g\bar{t}) &= (g, (a^{-1} \triangleleft g)(g\bar{s})a)
\end{aligned}$$

So we have $g\bar{t} = (a^{-1} \triangleleft g)(g\bar{s})a$ where $\bar{t}$ and $\bar{s}$ are principal parts of t and s. That means the set of arrows $Z^1(s,t)$ is bijective to set of elements $\{a \in A\}$ that satisfy $g\bar{t} = (a^{-1} \triangleleft g)(g\bar{s})a$.

Corollary 6.2.3. *The vertex group $Z^1\{s_1\}$ is isomorphic to the group A^G of elements of A fixed under the action of G.*

Proof. If we apply the previous argument to the set $Z^1\{s_1\}$,then we have

$$\begin{aligned} g\bar{s_1} &= (a^{-1} \triangleleft g)(g\bar{s_1})a \\ e_A &= (a^{-1} \triangleleft g)(e_A)a \\ e_A &= (a^{-1} \triangleleft g)a \end{aligned}$$

which means that elements of A fixed under G action are the only elements satisfying the last equation. That is $Z^1\{s_1\}$ isomorphic to the group A^G of elements of A fixed under G. $\square$

The set of elements of A fixed under the action of G, A^G, is sometimes written $H^0(G, A)$.

Example 6.2.4. Consider the vertex group $\mathsf{GPD}(\mathsf{G},\mathsf{G})\{1_\mathsf{G}\}$, a natural transformation from 1_G to 1_G is a conjugation by an element h of G and we find

$$\begin{aligned} (g)1_G &= h^{-1}(g)1_G h \\ g &= h^{-1}gh \end{aligned}$$

and that is satisfied only when $h \in ZG$, so $\mathsf{GPD}(\mathsf{G},\mathsf{G})\{1_\mathsf{G}\}$ is isomorphic to the center of the group G, denoted by ZG.

From a bijection between the set of sections of p and the set of principal parts of these sections we may consider $Der(G, A)$ as the vertex set of the groupoid $Z^1(G, A)$. Lemma 6.1.11 from the previous section tells that an arrow in $Z^1(d, f)$ exists if and only if

$$f = d * \chi_a.$$

this arrow labelled $a \in A$. Further, d and $(\chi_a * d)$ are connected by an arrow labelled $a\alpha_d$ in accordance with Lemma 6.1.12.

Cohomology and the Fundamental Exact Sequence

Definition Again following [3, section 5], the 1–dimensional cohomology set of G with coefficients in A is the set of components of the groupoid $Z^1(G, A)$:

$$H^1(G, A) = \pi_0 Z^1(G, A).$$

The component of the derivation d in $\pi_0 Z^1(G,A)$ is denoted by $[d]$.

Theorem 6.2.5. *The operation $*$ induces a monoid operation $\circ$ on the set $\pi_0 Z^1(G,A)$, such that the map $\varpi : Der(G,A) \to \pi_0 Z^1(G,A)$ given by $d \mapsto [d]$ is a surjective monoid morphism.*

Proof. It is sufficient to show that the equivalence relation $\simeq$ determined by ϖ:

$$d \simeq f \iff d \text{ and } f \text{ are connected in } Z^1(G,A)$$

is a congruence on $Der(G,A)$.
So suppose that d is connected to d' and f to f' with $a, b \in A$ such that

$$gd' = (a^{-1} \triangleleft g)(gd)a \text{ and } gf' = (b^{-1} \triangleleft g)(gf)b$$

Then by Lemma 6.1.11,

$$d' = d * \chi_a \text{ and } f' = f * \chi_b$$

and we have

$$\begin{aligned} d' * f' &= d * \chi_a * f * \chi_b \\ &= d * f * \chi_{a\alpha_f} * \chi_b \text{ , from Lemma 6.1.12} \\ &= d * f * \chi_{a\alpha_f b} \end{aligned}$$

and so $d' * f'$ is connected to $d * f$ by an arrow labelled $a\alpha_f b$. $\square$

Remark 6.2.6.

- The identity element of $\pi_0 Z^1(G,A)$ is the component $[\tau]$ of the trivial derivation $\tau : g \mapsto e_A$.
- The set of verticies of the component $[\tau]$ is the set of all inner derivations of $Der(G,A)$.
- Each component $[d]$ of $\pi_0 Z^1(G,A)$ is obtained from $[\tau]$ by left multiplication by d, that is $[\tau]$ has vertices $\{\chi_a : a \in A\}$ and the set of vertices of $[d]$ is $\{d * \chi_a : a \in A\}$.

So we now have $\pi_0 Z^1(G,A)$ as the first cohomology monoid $H^1(G,A)$.

Let

$$1 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ j\ } C \longrightarrow 1$$

be an exact sequence of crossed G-modules, then the fundamental exact sequence of non-abelian cohomology, as in [3], is the six term sequence of groups and based sets:

$$1 \longrightarrow A^G \xrightarrow{\ i\ } B^G \xrightarrow{\ j\ } C^G \xrightarrow{\ \partial\ } H^1(G,A) \xrightarrow{\ \bar{i}\ } H^1(G,B) \xrightarrow{\ \bar{j}\ } H^1(G,C)$$

But we have found that the last three terms of this sequence have monoid structure, so we want to make sure that ∂, $\bar{i}$ and $\bar{j}$ are monoid morphisms. Take $c \in C^G$ with $c = bj$ define $c\partial = [\chi_b]$ such that $\chi_b : g \mapsto (b^{-1} \triangleleft g)b$. The inner derivation χ_b can be considered as a derivation $G \to A$, for since $((b^{-1} \triangleleft g)b)j = c^{-1}c = e_C$ and by extactness of the sequence of crossed modules above $(b_{-1} \triangleleft g)b \in Ai$.
To show that ∂ is well-defined, suppose that $c = bj = b'j$ we need to show that χ_b is connected to $\chi_{b'}$ in $Z^1(G, A)$. By Lemma 6.1.11 this is equivalent to the existence of $a \in A$ such that

$$\begin{aligned}\chi_{b'} &= \chi_b * \chi_{ai}\\ &= \chi_{b(ai)}\end{aligned}$$

But $b^{-1}b' \in A_i$ since $b^{-1}b' \in \ker j$. So if $b^{-1}b' = ai$ then $b' = b(ai)$ as required.
Now if $c = e_C$ then $[\chi_b] \in H^1(G, A)$ is the identity element $[\tau]$. And if $c_1, c_2 \in C^G$ with $c_1 = b_1 j$ and $c_2 = b_2 j$ for some $b_1, b_2 \in B$ then by using Lemma 6.1.10 we find

$$\begin{aligned}(c_1\partial)(c_2\partial) &= [\chi_{b_1}] \circ [\chi_{b_2}]\\ &= [\chi_{b_1} * \chi_{b_2}]\\ &= [\chi_{b_1 b_2}]\\ &= (c_1c_2)\partial\end{aligned}$$

The proof will be the same for $\bar{i}$ and $\bar{j}$ so we are going for $\bar{i}$ only.We have $i : A \to B$ as a G-morphism induces $i' : Der(G, A) \to Der(G, B)$ such that $(gd)i' = (gd)i$, composition of d and i, Lemma 6.1.8 shows that i' is a monoid morphism.

Now, define $\bar{i} : \pi_0 Z^1(G, A) \to \pi_0 Z^1(G, B)$ such that $[d]\bar{i} = [di']$ and we aim to prove that $([d] \circ [f])\bar{i} = [d]\bar{i} \circ [f]\bar{i}$. We need first to make sure that $\bar{i}$ is well-defined, so let $f, d \in [d]$ that is $f = d * \chi_a$ then

$$\begin{aligned}fi' &= di' * \chi_a i'\\ &= di' * \chi_{ai}\end{aligned}$$

and that makes fi' and di' in the same component $[di']$,so $\bar{i}$ is well-defined.

$$\begin{aligned}([d] \circ [f])\bar{i} &= ([d * f])\bar{i}\\ &= [(d * f)i']\\ &= [di' * fi']\\ &= [di'] \circ [fi']\\ &= [d]\bar{i} \circ [f]\bar{i}.\end{aligned}$$

And

$$[\tau_A]\bar{i} = [\tau_A i'] = [\tau_B].$$

The Group $L^1(G, A)$

Let $Der^*(G, A)$ be the group of units of $Der(G, A)$. Then

Lemma 6.2.7. *The set of Inner derivations $Ider(G, A)$ is a normal subgroup of $Der^*(G, A)$.*

Proof. Let $d \in Der^*(G, A)$ and $\chi_a \in Ider(G, A)$, then from Lemma 6.1.12

$$d^{-1} * \chi_a * d = d^{-1} * d * \chi_{a\alpha_d} = \chi_{a\alpha_d} \in Ider(G, A).$$

□

In [24] Lue introduces the group $L^1(G, A)$ (although he used the notation H^1) as the quotient group $Der^*(G, A)/Ider(G, A)$, and we link this quotient group with the monoid $H^1(G, A)$ in the following lemma.

Lemma 6.2.8. *The group $L^1(G, A)$ can be considered as a subset of the monoid $H^1(G, A)$.*

Proof. For a unit $d \in Der^*(G, A)$, the set of vertices of the component $[d]$ is the coset $d * Ider(G, A) \in L^1(G, A)$. □

The following proposition gives J H C Whitehead's characterisation of the group $Der^*(G, A)$ as in [32].

Proposition 6.2.9. *The group $Der^*(G, A)$ characterised as follows:*

$d \in Der^*(G, A)$ *if and only if* θ_d *is an automorphism. Equivalently,*

$d \in Der^*(G, A)$ *if and only if* α_d *is an automorphism.*

Proof. Clearly, $d \in Der^*(G, A)$ implies that $\theta_d \in Aut(G)$ and that $\alpha_d \in Aut(A)$ since the map $\theta : Der(G, A) \to End(G)$ given by $d \mapsto \theta_d$ is a monoid morphism and it maps units $d \in Der^*(G, A)$ to units $\theta_d \in Aut(G)$. Similarly, for the map $\alpha : Der(G, A) \to End(A)$ given by $d \mapsto \alpha_d$.
Suppose now that $\theta_d \in Aut(G)$, the map $d' : G \to A$, defined by $gd' = (g\theta_d^{-1}d)^{-1}$, is a derivation and is the inverse of d. On the other hand, suppose that $\alpha_d \in Aut(A)$. Then $d'' : G \to A$ defined by $gd'' = (gd\alpha_d^{-1})^{-1}$, is a derivation and is the inverse of d. In either case we have $d \in Der^*(G, A)$. □

Corollary 6.2.10. *There are a group morphism $Der^*(G,A) \to Aut(A,G,\mu)$ given by $d \mapsto \langle \alpha_d, \theta_d \rangle$ with $\alpha_d \in Aut(A)$ and $\theta_d \in Aut(G)$, which satisfy*

$$\mu\theta_d = \alpha_d\mu \text{ and } (a \triangleleft g)\alpha_d = a\alpha_d \triangleleft g\theta_d.$$

Theorem 6.2.11. *$L^1(G,A)$ is the group of units of $H^1(G,A)$.*

Proof. In $H^1(G,A)$ we have defined

$$[d] \circ [f] = [d * f]$$

so, if $d \in Der^*(G,A)$ then by Lemma 6.2.8 , $[d] \in L^1(G,A)$ and d has inverse $\bar{d}$ such that $[d] \circ [\bar{d}] = [d * \bar{d}] = [\tau]$. That is $L^1(G,A) \subseteq$ group of units of $H^1(G,A)$.
For the converse, if $[d]$ is a unit in $H^1(G,A)$ with inverse $[f]$ then $[d * f] = [\tau]$, and $d * f$ is connected to τ. Hence $d * f * \chi_a = \tau$, and $f * \chi_a$ is the right inverse of d, and $[f] = [f * \chi_a]$.
If we put $k = f * \chi_a$ we have $d * k = \tau$, and then $\alpha_{d*k} = \alpha_d\alpha_k = 1_A$ so α_d is injective.
But $[k]$ is also a left inverse for $[d]$ in $H^1(G,A)$, so $k * d = \chi_b$ for some $b \in A$. Therefore, $\alpha_k\alpha_d = \alpha_{\chi_b}$ which is a conjugation by b and for that we can write $\alpha_{\chi_b}^{-1}\alpha_k\alpha_d = 1_A$ so α_d is surjective.
we find that if $[d]$ is a unit in $H^1(G,A)$, then α_d is bijective and from Proposition 6.2.9 we find $d \in Der^*(G,A)$. $\square$

The following discussion about the exact sequence of groups is detailed by Lue in [24]. Consider the crossed G-modules $\mu : A \to G$, $\kappa : \ker\mu \to G$ and $i : A\mu \hookrightarrow G$, where G action in κ is induced by $w : G \to Aut(A)$. Each $d \in Der^*(G,A)$ gives rise to an element $d\mu \in Der^*(G,A\mu)$, and we note that $d\mu \in Ider(G,A\mu)$ if and only if, for some $a \in A$, we have $(\chi_a * d)\mu = 0$. From this we have the exact sequence of groups

$$L^1(G,\ker\mu) \to L^1(G,A) \to L^1(G,A\mu).$$

If we define $A^G = \{a \in A : a \triangleleft g = a \text{ for all } g \in G\}$, then we have the six terms exact sequence of groups

$$1 \to (\ker\mu)^G \to A^G \to (A\mu)^G \to L^1(G,\ker\mu) \to L^1(G,A) \to L^1(G,A\mu).$$

6.3 $\langle\psi,\phi\rangle$–derivations

In this section we try to generalize our work on derivations by studying the case of $\langle\psi,\phi\rangle$-derivations in which $\langle\psi,\phi\rangle$ is an endomorphism of the crossed module $\mu : A \to G$. Then we construct the monoid $(Der_\diamond(G,A),*)$ of all derivations linked to endomorphisms of μ. After that, we use this monoid as the arrow set to construct the groupoid $Gpd_\diamond(\mu,\mu)$ with set of objects as $\mathrm{End}(A, G, \mu)$. Finally, we verify that this groupoid is a monoid in the category of groupoids by satisfying the Eckmann-Hilton condition. The monoid $Gpd_\diamond(\mu, \mu)$ has been discussed before: it is the groupoid $\mathcal{E}(A, G)$ of [28], and is also the groupoid $\underline{\mathsf{END}}(A, G)$ of [14], in which its structure is obscurely derived from the cartesian closed structure on the category of groupoids. Our aim here is to give a self-contained and transparent acccount.

The monoid $Der_\diamond(G, A)$

Given a crossed module $\mu : A \to G$ over a group G and a group endomorphism $\phi : G \to G$, a ϕ-derivation $d : G \to A$ is a map satisfying

$$(gh)d = (gd \triangleleft h\phi)hd \text{ , for all } g, h \in G.$$

We now consider the case in which ϕ is the G-component of an endomorphism $\langle \psi, \phi \rangle$ of the crossed module μ. Hence we have the commutative square

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[d, "\psi"'] & G \arrow[d, "\phi"] \\ A \arrow[r, "\mu"'] & G \end{tikzcd}$$

and the relation $(a \triangleleft g)\psi = a\psi \triangleleft g\phi$ is satisfied for all $a \in A$ and $g \in G$. We say that d is a $\langle \psi, \phi \rangle$-derivation, and that d is linked to $\langle \psi, \phi \rangle$.

Lemma 6.3.1. *Although d is not a group morphism, we have $(e_G)d = e_A$.*

Proof. If we put $h = e_G$ in the defining property above, then we find that

$$\begin{aligned} (g(e_G))d &= gd \triangleleft (e_G)\phi.(e_G)d \\ gd &= gd.(e_G)d \\ (gd)^{-1}.gd &= (e_G)d \\ e_A &= (e_G)d. \end{aligned}$$

□

Remark 6.3.2.
- The defining property of d above only involves ϕ, but further properties will involve both components of the endomorphism $\langle \psi, \phi \rangle$.
- We shall include the linked endomorphism $\langle \psi, \phi \rangle$ in the definition of d since the same map d could be linked to more than one endomorphism.

Definition Given a crossed module $\mu : A \to G$ with an endomorphism $\langle \psi, \phi \rangle : \mu \to \mu$. A $\langle \psi, \phi \rangle$-derivation $d : G \to A$ is a map that satisfies,for all $g, h \in G$

$$(gh)d = (gd \triangleleft h\phi)hd$$

The set of all such derivations is denoted by $Der_{\langle\psi,\phi\rangle}(G, A)$, and d determines two endomorphisms

$$\theta_d : G \to G \text{ defined by } g\theta_d = (g\phi)(gd\mu), \text{ and}$$

$$\alpha_d : A \to A \text{ defined by } a\alpha_d = (a\psi)(a\mu d).$$

Lemma 6.3.3. *The pair $\langle\alpha_d, \theta_d\rangle$ is an endomorphism of $\mu : A \to G$.*

Proof. We first check that $\mu\theta_d = \alpha_d\mu$, for all $a \in A$

$$\begin{aligned}
a\mu\theta_d &= (a\mu\phi)(a\mu d\mu) \\
&= (a\psi\mu)(a\mu d\mu) \\
&= ((a\psi)(a\mu d))\mu \\
&= a\alpha_d\mu.
\end{aligned}$$

And we have

$$\begin{aligned}
(a \triangleleft g)\alpha_d &= (a \triangleleft g)\psi(a \triangleleft g)\mu d \\
&= a\psi \triangleleft g\phi(g^{-1}(a\mu)g)d \\
&= a\psi \triangleleft g\phi(g^{-1}d) \triangleleft a\mu\phi \triangleleft g\phi((a\mu)g)d \\
&= a\psi \triangleleft g\phi(g^{-1}d) \triangleleft a\psi\mu \triangleleft g\phi[(a\mu d) \triangleleft g\phi]gd \\
&= a\psi \triangleleft g\phi((a\psi)^{-1}g^{-1}d(a\psi)) \triangleleft g\phi[(a\mu d) \triangleleft g\phi]gd \\
&= [(a\psi(a\psi)^{-1}(g^{-1}d)(a\psi)(a\mu d)) \triangleleft g\phi]gd \\
&= [((g^{-1}d)(a\psi)(a\mu d)) \triangleleft g\phi]gd \\
&= [((gd)^{-1} \triangleleft (g\phi)^{-1}a\psi a\mu d) \triangleleft g\phi]gd \\
&= (gd)^{-1}((a\psi a\mu d) \triangleleft g\phi)gd \\
&= ((a\psi a\mu d) \triangleleft g\phi) \triangleleft gd\mu \\
&= (a\psi a\mu d) \triangleleft g\phi gd\mu \\
&= (a\alpha_d) \triangleleft g\theta_d.
\end{aligned}$$

□

We now define a composition of any two derivations. Let d be a $\langle\psi, \phi\rangle$-derivation, and f a $\langle\sigma, \rho\rangle$-derivation. We define the composition $*$ by

$$g(d * f) = (gd\sigma)(g\theta_d f).$$

As for the composition of derivations considered earlier which are linked to $\langle 1_A, 1_G\rangle$, we have an alternative expression for $*$ using ϕ and α_f.

Lemma 6.3.4. $g(d * f) = (g\phi f)(gd\alpha_f)$.

Proof.

$$\begin{aligned}
g(d * f) &= (gd\sigma)((g\phi)(gd\mu))f \\
&= (gd\sigma)[(g\phi f) \triangleleft (gd\mu\rho)](gd\mu f) \\
&= (gd\sigma)[(g\phi f) \triangleleft (gd\sigma\mu)](gd\mu f) \\
&= (gd\sigma)(gd\sigma)^{-1}(g\phi f)(gd\sigma)(gd\mu f) \\
&= (g\phi f)(gd\sigma)(gd\mu f) \\
&= (g\phi f)(gd\alpha_f).
\end{aligned}$$

□

Theorem 6.3.5. *Let $d : G \to A$ be a $\langle\psi, \phi\rangle$-derivation, and $f : G \to A$ a $\langle\sigma, \rho\rangle$-derivation. Then $d * f$ is a $\langle\psi\sigma, \phi\rho\rangle$-derivation, with $\theta_{d*f} = \theta_d\theta_f$ and $\alpha_{d*f} = \alpha_d\alpha_f$.*

Proof.

$$\begin{aligned}
(gh)(d * f) &= (gh)d\sigma(gh)\theta_d f \\
&= ((gd \triangleleft h\phi)hd)\sigma(g\theta_d f \triangleleft h\theta_d\rho)(h\theta_d f) \\
&= (gd \triangleleft h\phi)\sigma(hd\sigma)(g\theta_d f \triangleleft h\theta_d\rho)(h\theta_d f) \\
&= gd\sigma \triangleleft (h\phi\rho)(hd\sigma)[(g\theta_d f) \triangleleft ((h\phi)(hd\mu))\rho](h\theta_d f) \\
&= gd\sigma \triangleleft (h\phi\rho)(hd\sigma)[(g\theta_d f) \triangleleft (h\phi\rho)(hd\mu\rho)](h\theta_d f) \\
&= gd\sigma \triangleleft (h\phi\rho)(hd\sigma)[(g\theta_d f) \triangleleft (h\phi\rho)(hd\sigma\mu)](h\theta_d f) \\
&= gd\sigma \triangleleft (h\phi\rho)(hd\sigma)(hd\sigma)^{-1}[(g\theta_d f) \triangleleft (h\phi\rho)](hd\sigma)(h\theta_d f) \\
&= gd\sigma \triangleleft (h\phi\rho)[(g\theta_d f) \triangleleft (h\phi\rho)](hd\sigma)(h\theta_d f) \\
&= [(gd\sigma)(g\theta_d f) \triangleleft (h\phi\rho)](hd\sigma)(h\theta_d f) \\
&= [g(d * f) \triangleleft (h\phi\rho)]h(d * f)
\end{aligned}$$

and so $d * f$ is a $\langle\psi\sigma, \phi\rho\rangle$-derivation.
Now, we have

$$g\theta_{d*f} = (g\phi\rho)[(gd\sigma)(g\phi.gd\mu)f]\mu = (g\phi\rho)(gd\sigma\mu)(g\phi.gd\mu)f\mu,$$

whilst

$$g\theta_d\theta_f = (g\phi.gd\mu)\theta_f = (g\phi\rho)(gd\mu\rho)(g\phi.gd\mu)f\mu.$$

Since $\sigma\mu = \mu\rho$, we see that $\theta_{d*f} = \theta_d\theta_f$.
Similarly,

$$a\alpha_{d*f} = (a\psi\sigma)((a\mu)d * f) = (a\psi\sigma)(a\mu d\sigma)(a\mu\theta_d f) = (a\psi a\mu d)\sigma[(a\mu\phi)(a\mu d\mu)]f$$

while

$$a\alpha_d\alpha_f = (a\psi a\mu d)\alpha_f = (a\psi a\mu d)\sigma(a\psi a\mu d)\mu f = (a\psi a\mu d)\sigma[(a\psi\mu)(a\mu d\mu)]f\,.$$

□

With these facts established, we can show that $*$ is associative on the set $Der_\diamond(G, A)$ of all derivations $G \to A$ linked to endomorphisms of the crossed module $\mu : A \to G$.

Theorem 6.3.6. *The set $Der_\diamond(G, A)$ is a monoid under the operation $*$.*

Proof. Let d and f be as above, and let k be a $\langle\kappa, \eta\rangle$-derivation, then

$$\begin{aligned} g((d * f) * k) &= (g(d * f)\kappa)(g\theta_{d*f}k) \\ &= [(gd\sigma)(g\theta_d f)]\kappa(g\theta_d\theta_f k) \\ &= (gd\sigma\kappa)(g\theta_d f\kappa)(g\theta_d\theta_f k) \\ &= (gd\sigma\kappa)[(g\theta_d)(f * k)] \\ &= g(d * (f * k)) \end{aligned}$$

so, $*$ is associative. And the trivial derivation τ which is linked to $\langle 1_A, 1_G\rangle$ is the identity element of the set $Der_\diamond(G, A)$ since
$g(d * \tau) = (gd)1_A(g\theta_d)\tau = gd$ and $g(\tau * d) = (g\tau)\psi(g)1_G d = gd$. □

Remark 6.3.7. We should think of elements of $Der_\diamond(G, A)$ as pairs $(d, \langle\psi, \phi\rangle)$ since the same derivation d could be linked to more than one endomorphism.

We now discuss the group of units in $Der_\diamond(G, A)$. Recall that:

- If d is a $\langle\psi, \phi\rangle$-derivation and f a $\langle\sigma, \rho\rangle$-derivation then $d*f$ is a $\langle\psi\sigma, \phi\rho\rangle$-derivation.
- If $f = d^{-1}$ (d is a unit in $Der_\diamond(G, A)$) then $d*f$ is the identity element of $Der_\diamond(G, A)$ which is the trivial derivation $(\tau, \langle 1_A, 1_G\rangle)$. So $\psi = \sigma^{-1}$ and $\phi = \rho^{-1}$.
- $\alpha_{d*f} = \alpha_d\alpha_f$ and $\theta_{d*f} = \theta_d\theta_f$. So if $f = d^{-1}$ then we have $\alpha_\tau = 1_A$ and $\theta_\tau = 1_G$ that is $\alpha_d = \alpha_f^{-1}$ and $\theta_d = \theta_f^{-1}$.
- Thus, we must have $\langle\psi, \phi\rangle$ and $\langle\alpha_d, \theta_d\rangle$ as automorphisms of the crossed module μ.

Let d be a unit and $f = d^{-1}$ then

$$\begin{aligned} g(d * f) &= (gd\psi^{-1})(g\theta_d f) \\ e_A &= (gd\psi^{-1})(g\theta_d f) \end{aligned}$$

so we have $g\theta_d f = (gd\psi^{-1})^{-1}$. And

$$\begin{aligned} gf &= g\theta_d^{-1}\theta_d f \\ &= (g\theta_d^{-1} d\psi^{-1})^{-1} \end{aligned}$$

The groupoid $Gpd_\diamond(\mu, \mu)$

Let $\mu : A \to G$ be a crossed module over the group G, $\langle \psi, \phi \rangle$ an endomorphism of μ, and d a derivation linked to $\langle \psi, \phi \rangle$. As we have seen d induces an endomorphism of μ, namely $\langle \alpha_d, \theta_d \rangle$. We shall define a groupoid $Gpd_\diamond(\mu, \mu)$ with vertex set $\mathrm{End}(A, G, \mu)$ and arrow set $Der_\diamond(G, A)$.
A $\langle \psi, \phi \rangle$-derivation $d \in Der_\diamond(G, A)$ will be an arrow from $\langle \psi, \phi \rangle$ to $\langle \alpha_d, \theta_d \rangle$. Now, let f be an $\langle \alpha_d, \theta_d \rangle$-derivation and define the composition $d \cdot f$ to be just pointwise product of values in A:

$$g(d \cdot f) = (gd)(gf)$$

and before checking the groupoid axioms, we need the following lemmas

Lemma 6.3.8. *The derivation $d \cdot f$ is linked to $\langle \psi, \phi \rangle$.*

Proof.

$$\begin{aligned} (gh)(d \cdot f) &= (gh)d(gh)f \\ &= (gd \triangleleft h\phi)hd(gf \triangleleft h\theta_d)hf \\ &= (gd \triangleleft h\phi)hd[gf \triangleleft (h\phi)(hd\mu)]hf \\ &= (gd \triangleleft h\phi)hd(hd)^{-1}(gf \triangleleft h\phi)hdhf \\ &= (gd)(gf) \triangleleft h\phi(hd)(hf) \\ &= (g(d \cdot f) \triangleleft h\phi)h(d \cdot f) \end{aligned}$$

so $(d \cdot f)$ is a $\langle \psi, \phi \rangle$ derivation. □

Lemma 6.3.9. $\alpha_{d \cdot f} = \alpha_f$ *and* $\theta_{d \cdot f} = \theta_f$.

Proof.

$$\begin{aligned}
a\alpha_{d\cdot f} &= (a\psi)((a\mu)(d\cdot f)) \\
&= (a\psi)(a\mu d)(a\mu f) \\
&= ((a\psi)(a\mu d))(a\mu f) \\
&= (a\alpha_d)(a\mu f) \\
&= a\alpha_f
\end{aligned}$$

$$\begin{aligned}
g\theta_{d\cdot f} &= g\phi(g(d\cdot f))\mu \\
&= g\phi((gd)(gf))\mu \\
&= ((g\phi)(gd\mu))(gf\mu) \\
&= (g\theta_d)(gf\mu) \\
&= g\theta_f
\end{aligned}$$

□

Since $\cdot$ is just product of values in the group A, clearly it is associative. And $d\cdot\tau = \tau\cdot d = d$ gives an identity element at each vertex $\langle\psi,\phi\rangle$. Hence,

Theorem 6.3.10. *The set of objects* $\mathrm{End}(A,G,\mu)$ *with the set of arrows* $Der_{\diamond}(G,A)$ *form a category.*

To find out if this category is a groupoid, let d be a $\langle\psi,\phi\rangle$-derivation and let d^{-1} be the derivation linked to $\langle\alpha_d,\theta_d\rangle$ defined as

$$(g)d^{-1} = (gd)^{-1}$$

then we have,

$$g(d\cdot d^{-1}) = (gd)(gd)^{-1} = e_A = (gd)^{-1}(gd) = g(d^{-1}\cdot d).$$

To say that d^{-1} is the inverse of d we need to prove the following lemmas.

Lemma 6.3.11. d^{-1} *is linked to* $\langle\alpha_d,\theta_d\rangle$.

Proof.

$$\begin{aligned}(gh)d^{-1} &= ((gh)d)^{-1}\\ &= ((gd \triangleleft h\phi)hd)^{-1}\\ &= (hd)^{-1}(gd \triangleleft h\phi)^{-1}\\ &= (hd)^{-1}((gd)^{-1} \triangleleft h\phi)\\ &= (hd)^{-1}((gd)^{-1} \triangleleft h\phi)(hd)(hd)^{-1}\\ &= [(gd)^{-1} \triangleleft (h\phi)(hd\mu)](hd)^{-1}\\ &= (gd^{-1} \triangleleft h\theta_d)hd^{-1}\end{aligned}$$

□

Lemma 6.3.12. $\langle \alpha_{d^{-1}}, \theta_{d^{-1}} \rangle = \langle \psi, \phi \rangle$.

Proof.

$$\begin{aligned}a\alpha_{d^{-1}} &= (a\alpha_d)(a\mu d^{-1})\\ &= a\psi(a\mu d)(a\mu d)^{-1}\\ &= a\psi\end{aligned}$$

$$\begin{aligned}g\theta_{d^{-1}} &= (g\theta_d)(gd^{-1}\mu)\\ &= g\phi(gd\mu)(gd^{-1}\mu)\\ &= g\phi[(gd)(gd)^{-1}]\mu\\ &= g\phi\end{aligned}$$

□

Theorem 6.3.13. *The category with vertex set $End(A, G, \mu)$ and arrow set $Der_\diamond(G, A)$ is a groupoid denoted by $Gpd_\diamond(\mu, \mu)$.*

The group $Gpd_\diamond(\mu, \mu)\{\langle 1_A, 1_G \rangle\}$

In the groupoid $Gpd_\diamond(\mu, \mu)$ the vertex group at $\langle 1_A, 1_G \rangle$ is as follows

$$Gpd_\diamond(\mu, \mu)\{\langle 1_A, 1_G \rangle\} = \{d : d \text{ is a derivation such that } \alpha_d = 1_A \text{ and } \theta_d = 1_G\}.$$

Meaning,

1. $a\alpha_d = a(a\mu d) = a$ implies that $d : Im(\mu) \to e_A$.

2. $g\theta_d = g(gd\mu) = g$ implies that $d : G \to \ker(\mu)$.

Lemma 6.3.14. *The group $Gpd_\diamond(\mu, \mu)\{\langle 1_A, 1_G\rangle\}$ is abelian.*

Proof. If $d, f \in Gpd_\diamond(\mu, \mu)\{\langle 1_A, 1_G\rangle\}$ then d and f map G to $\ker(\mu)$ and since $\ker(\mu)$ is central we have $(gd)(gf) = (gf)(gd)$ for all $g \in G$ and so the group $Gpd_\diamond(\mu, \mu)\{\langle 1_A, 1_G\rangle\}$ is abelian. □

Remark 6.3.15. If $Q = G/Im(\mu)$ then $\ker(\mu)$ is a Q-module, since $Im(\mu)$ acts trivially on $\ker(\mu)$.

Proposition 6.3.16. *Given $d \in Gpd_\diamond(\mu, \mu)\{\langle 1_A, 1_G\rangle\}$ and define $F : Gpd_\diamond(\mu, \mu)\{\langle 1_A, 1_G\rangle\} \to Der(Q, \ker(\mu))$ where $(d : G \to \ker(\mu)) \mapsto (\tilde{d} : Q \to \ker(\mu))$ such that $x(Im(\mu))\tilde{d} = xd$. Then $\tilde{d}$ is a well defined derivation*

Proof. We have

$$(x.a\mu)d = (xd \triangleleft a\mu)(a\mu d) = xd,$$

and

$$\begin{aligned}(pq)\tilde{d} &= ((x.Im\mu)(y.Im\mu))\tilde{d} \\ &= (xy.Im\mu)\tilde{d} \\ &= (xy)d \\ &= (xd \triangleleft y)yd \\ &= [(x.Im\mu)\tilde{d} \triangleleft (y.Im\mu)](y.Im\mu)\tilde{d} \\ &= (p\tilde{d} \triangleleft q)q\tilde{d}\end{aligned}$$

□

Corollary 6.3.17. *Since $ker(\mu)$ is an abelian group, we have $Der(Q, \ker\mu)$ as an abelian group with identity element $\tilde{\tau}$ and an inverse of $\tilde{d}$ is $\tilde{d}^{-1}$ defined as $q\tilde{d}^{-1} = xd^{-1}$ such that $q = x.Im\mu$.*

Theorem 6.3.18. *The mapping F is an isomorphism.*

Proof. Let $d, f \in Gpd_\diamond(\mu, \mu)\{\langle 1_A, 1_G\rangle\}$ such that $dF = fF$ that is $q\tilde{d} = q\tilde{f}$ for each $q \in Q$ and from the definition of $\tilde{d}$, we have $xd = xf$ for each $x \in G$ which means $d = f$ and that proves the injectivity.
Let $\varpi : G \to Q$ be the quotient mapping, and consider a derivation $\tilde{d} : Q \to \ker\mu$, then $d = \varpi\tilde{d} : G \to \ker\mu$ and it is a derivation, remember that g and $(g\varpi)$ act on $\ker\mu$ in the same way, so
$(gh)d = [(g\varpi)(h\varpi)]\tilde{d} = (g\varpi\tilde{d} \triangleleft h\varpi)h\varpi\tilde{d} = (gd \triangleleft h)hd.$

Now the mapping $\tilde{d} \mapsto d$ is a mapping $Der(Q, \ker\mu) \to Der(G, \ker\mu)$ and it is obviously injective, since if $d = f$ then $g\varpi\tilde{d} = g\varpi\tilde{f}$ for all $g \in G$ and so $\tilde{d} = \tilde{f}$. Moreover, $(a\mu)d = (a\mu\varpi)\tilde{d} = (e_Q)\tilde{d} = e_A$ and so $d \in Gpd_\diamond(\mu,\mu)\{1_A, 1_G\}$. But then F maps $d \mapsto \tilde{d}$ and so it is surjective.
This shows that $Gpd_\diamond(\mu,\mu)\{1_A, 1_G\}$ is isomorphic to $Der(Q, \ker\mu)$. $\square$

A crossed module $\mu : A \to G$ induces another crossed module $\Delta : Der^*(G, A) \to Aut(A, G, \mu)$ given by $d \mapsto \langle \alpha_d, \theta_d \rangle$, as shown in [24]. In the following theorem we link the group $Gpd_\diamond(\mu,\mu)\{\langle 1_A, 1_G\rangle\}$ and the crossed module Δ.

Theorem 6.3.19. $Gpd_\diamond(\mu,\mu)\{\langle 1_A, 1_G\rangle\} = \ker\Delta$.

Proof. We have $\ker\Delta = \{d \in Der^*(G, A) : d\Delta = \langle 1_A, 1_G\rangle\}$, so it is clear that

$$Gpd_\diamond(\mu,\mu)\{\langle 1_A, 1_G\rangle\} \subseteq \ker\Delta.$$

To prove the converse, let $d \in \ker\Delta$, that is d is a derivation and a unit such that $d\Delta = \langle 1_A, 1_G\rangle$. Since d is a derivation it is linked to $\langle 1_A, 1_G\rangle$, and we can say that d is a loop over $\langle 1_A, 1_G\rangle$, that is $d \in Gpd_\diamond(\mu,\mu)\{\langle 1_A, 1_G\rangle\}$ implies $\ker\Delta \subseteq Gpd_\diamond(\mu,\mu)\{\langle 1_A, 1_G\rangle\}$. $\square$

The groupoid $Gpd_\diamond(\mu,\mu)$ is monoidal

The purpose of this discussion is to show that the groupoid $Gpd_\diamond(\mu,\mu)$ is a monoid in the category of groupoids, starting with a relation between the operations in the monoid $Der_\diamond(G, A)$ and the groupoid $Gpd_\diamond(\mu,\mu)$, and using the Eckmann-Hilton condition eventually. Let d be a $\langle\psi, \phi\rangle$-derivation, and f a $\langle\sigma, \rho\rangle$-derivation. As we know

$$d * f = (d\sigma)(\theta_d f) = (\phi f)(d\alpha_f)$$

The composition $d * f$ in the monoid $Der_\diamond(G, A)$ can be displayed as a composition of two elements in the groupoid $Gpd_\diamond(\mu,\mu)$ in two ways, as shown in the following square

$$\begin{tikzcd}
\langle \psi\sigma, \phi\rho \rangle \arrow[r, "\ddot{f}"] \arrow[d, "\dot{d}"'] \arrow[dr, "d*f"] & \langle \alpha_{\ddot{f}}, \theta_{\ddot{f}} \rangle \arrow[d, "\ddot{d}"] \\
\langle \alpha_{\dot{d}}, \theta_{\dot{d}} \rangle \arrow[r, "\dot{f}"'] & \langle \alpha_{d*f}, \theta_{d*f} \rangle
\end{tikzcd}$$

where $\dot{d} = d\sigma$, $\dot{f} = \theta_d f$ and $\ddot{d} = d\alpha_f$, $\ddot{f} = \phi f$.
Now, we prove the commutativity of the square starting with $\dot{d} = d\sigma : G \to A \to A$, it

should be a $\langle \psi\sigma, \phi\rho \rangle$-derivation, so for g and h in G we have

$$\begin{aligned}(gh)\dot{d} &= (gh)d\sigma \\ &= (gd \triangleleft h\phi)\sigma(hd)\sigma \\ &= (gd\sigma \triangleleft h\phi\rho)hd\sigma\end{aligned}$$

We also have $\alpha_{\dot{d}} = (\psi\sigma)(\mu d\sigma)$ and $\theta_{\dot{d}} = (\phi\rho)(d\sigma\mu)$.
Moving to $\dot{f} = \theta_d f : G \to G \to A$, it is an $\langle \alpha_{\dot{d}}, \theta_{\dot{d}} \rangle$-derivation since

$$\begin{aligned}(gh)\dot{f} &= (gh)\theta_d f \\ &= (g\theta_d h\theta_d)f \\ &= (g\theta_d f \triangleleft h\theta_d\rho)h\theta_d f \\ &= [g\theta_d f \triangleleft (h\phi hd\mu)\rho]h\theta_d f \\ &= (g\theta_d f \triangleleft h\phi\rho hd\mu\rho)h\theta_d f \\ &= g\theta_d f \triangleleft (h\phi\rho hd\sigma\mu)h\theta_d f \\ &= (g\theta_d f \triangleleft h\theta_{d\sigma})h\theta_d f \\ &= (g\dot{f} \triangleleft h\theta_{\dot{d}})h\dot{f}\end{aligned}$$

We are left now with proving that $\langle \alpha_{\dot{f}}, \theta_{\dot{f}} \rangle = \langle \alpha_{d*f}, \theta_{d*f} \rangle$, so for $a \in A$ we have

$$\begin{aligned}a\alpha_{\dot{f}} &= (a\alpha_{\dot{d}})(a\mu\dot{f}) \\ &= (a\alpha_{d\sigma})(a\mu\theta_d f) \\ &= (a\psi\sigma)(a\mu d\sigma)(a\mu\theta_d f) \\ &= (a\psi\sigma)(a\mu d\sigma)[(a\mu\phi)(a\mu d\mu)]f \\ &= (a\psi a\mu d)\sigma[(a\psi)(a\mu d)]\mu f \\ &= (a\alpha_d)\sigma(a\alpha_d)\mu f \\ &= (a\alpha_d)\alpha_f \\ &= a\alpha_{d*f}\end{aligned}$$

Similarly, for all $g \in G$ we have

$$\begin{aligned}g\theta_{\dot{f}} &= g\theta_{\theta_d f} \\ &= (g\theta_{\dot{d}})(g\theta_d f\mu) \\ &= (g\phi\rho)(gd\sigma\mu)(g\phi gd\mu)f\mu \\ &= (g\phi gd\mu)\rho(g\phi gd\mu)f\mu \\ &= (g\theta_d)\theta_f \\ &= g\theta_{d*f}\end{aligned}$$

We have proved the commutativity of the lower triangle in the above square, that is $d * f = \dot{d} \cdot \dot{f}$. And to prove it for the upper triangle, $d * f = \ddot{f} \cdot \ddot{d}$, we consider the equivalent expression of the composition $d * f = (\phi f)(d\alpha_f)$, and follow the same steps above.
Start with linking $\ddot{f} = \phi f : G \to G \to A$ to $\langle \psi\sigma, \phi\rho \rangle$ since we have
$(gh)\ddot{f} = (gh)\phi f = (g\phi f \triangleleft h\phi\rho)h\phi f.$
Then define $\alpha_{\ddot{f}}$ and $\theta_{\ddot{f}}$ as follows
$\alpha_{\ddot{f}} = \alpha_{\phi f} = (\psi\sigma)(\mu\phi f)$ and $\theta_{\ddot{f}} = \theta_{\phi f} = (\phi\rho)(\phi f \mu)$.
We have $\ddot{d} = d\alpha_f : G \to A \to A$ as an $\langle \alpha_{\ddot{f}}, \theta_{\ddot{f}} \rangle$-derivation satisfies

$$\begin{aligned}
(gh)\ddot{d} &= (gh)d\alpha_f \\
&= (gd \triangleleft h\phi)\alpha_f (hd)\alpha_f \\
&= (gd\alpha_f \triangleleft h\phi\theta_f)hd\alpha_f \\
&= (gd\alpha_f \triangleleft h\phi\rho h\phi f\mu)hd\alpha_f \\
&= (gd\alpha_f \triangleleft h\theta_{\ddot{f}})hd\alpha_f \\
&= (g\ddot{d} \triangleleft h\theta_{\ddot{f}})h\ddot{d}
\end{aligned}$$

Again we are left with proving that $\langle \alpha_{\ddot{d}}, \theta_{\ddot{d}} \rangle = \langle \alpha_{d*f}, \theta_{d*f} \rangle$, so for $a \in A$ we have

$$\begin{aligned}
a\alpha_{\ddot{d}} &= a\alpha_{d\alpha_f} \\
&= (a\alpha_{\ddot{f}})(a\mu\ddot{d}) \\
&= (a\alpha_{\phi f})(a\mu d\alpha_f) \\
&= (a\psi\sigma)(a\mu\phi f)(a\mu d\sigma)(a\mu d\mu)f \\
&= (a\psi\sigma)(a\mu d\sigma)(a\mu d\sigma)^{-1}(a\mu\phi f)(a\mu d\sigma)(a\mu d\mu f) \\
&= (a\psi\sigma)(a\mu d\sigma)[a\mu\phi f \triangleleft a\mu d\sigma\mu](a\mu d\mu f) \\
&= (a\psi\sigma)(a\mu d\sigma)[a\mu\phi f \triangleleft a\mu d\mu\rho](a\mu d\mu f) \\
&= (a\psi\sigma)(a\mu d\sigma)[(a\mu\phi)(a\mu d\mu)]f \\
&= (a\psi\sigma)(a\mu d\sigma)(a\mu\theta_d f) \\
&= a\alpha_{d*f}
\end{aligned}$$

And for all $g \in G$ we also have

$$\begin{aligned}
g\theta_{\ddot{d}} &= (g\theta_{\ddot{f}})(g\ddot{d}\mu) \\
&= (g\theta_{\phi f})(gd\alpha_f\mu) \\
&= (g\phi\rho)(g\phi f\mu)(gd\sigma\mu)(gd\mu f\mu) \\
&= (g\phi\rho)(gd\sigma\mu)(gd\mu\rho)^{-1}(g\phi f\mu)(gd\mu\sigma)(gd\mu f\mu) \\
&= (g\phi\rho)(gd\sigma\mu)(g\phi f \triangleleft gd\mu\rho)\mu(gd\mu f\mu) \\
&= (g\phi\rho)(gd\sigma\mu)[(g\phi f \triangleleft gd\mu\rho)gd\mu f]\mu \\
&= (g\phi\rho)(gd\sigma\mu)[(g\phi.gd\mu)f]\mu \\
&= (g\phi gd\mu)\rho(g\phi gd\mu)f\mu \\
&= g\theta_{d*f}
\end{aligned}$$

Back to the groupoid $Gpd_{\diamond}(\mu,\mu)$, Let d, k, f, and l be elements in $Gpd_{\diamond}(\mu,\mu)$ such that d is linked to $\langle\psi,\phi\rangle$ and k to $\langle\sigma,\rho\rangle$ and $d\cdot f$, $k\cdot l$ exist. If these four elements satisfy the Eckmann-Hilton condition:

$$(d * k)\cdot(f * l) = (d\cdot f) * (k\cdot l)$$

$$\begin{tikzcd}
\langle\psi,\phi\rangle \arrow[d, "d"'] \arrow[dr, "d\cdot f"] & \\
\langle\alpha_d,\theta_d\rangle \arrow[r, "f"'] & \langle\alpha_f,\theta_f\rangle
\end{tikzcd}$$

$$\begin{tikzcd}
\langle\sigma,\rho\rangle \arrow[d, "k"'] \arrow[dr, "k\cdot l"] & \\
\langle\alpha_k,\theta_k\rangle \arrow[r, "l"'] & \langle\alpha_l,\theta_l\rangle
\end{tikzcd}$$

then the groupoid $Gpd_{\diamond}(\mu,\mu)$ is a monoid in the category of groupoids. That is the composition $*$, defined on objects as $\langle\psi,\phi\rangle * \langle\sigma,\rho\rangle = \langle\psi\sigma,\phi\rho\rangle$ and on elements as $d * f = \dot{d}\cdot\dot{f} =$

$\ddot{f} \cdot \ddot{d}$, is a functor $(Gpd_\diamond(\mu,\mu)) \times (Gpd_\diamond(\mu,\mu)) \to Gpd_\diamond(\mu,\mu)$

$$\begin{aligned}
g[(d*k)\cdot(f*l)] &= g(d\sigma\theta_d k)g(f\alpha_k\theta_f l) \\
&= (gd\sigma)(g\theta_d k)(gf\alpha_k)(g\theta_f l) \\
&= (gd\sigma)(g\theta_d k)(gf\alpha_k)(g\theta_d gf\mu)l \\
&= (gd\sigma)(g\theta_d k)(gf\alpha_k)(g\theta_d l \triangleleft gf\mu\theta_k)(gf\mu l) \\
&= (gd\sigma)(g\theta_d k)(gf\alpha_k)(g\theta_d l \triangleleft gf\alpha_k\mu)(gf\mu l) \\
&= (gd\sigma)(g\theta_d k)(gf\alpha_k)(gf\alpha_k)^{-1}(g\theta_d l)(gf\alpha_k)(gf\mu l) \\
&= (gd\sigma)(g\theta_d k)(g\theta_d l)(gf\alpha_k)(gf\mu l) \\
&= (gd\sigma)(gf\sigma)(gf\sigma)^{-1}(g\theta_d k)(gf\alpha_k)(gf\alpha_k)^{-1}(g\theta_d l)(gf\alpha_k)(gf\mu l) \\
&= (gd\sigma)(gf\sigma)(g\theta_d k \triangleleft gf\sigma\mu)(gf\mu k)(g\theta_d l \triangleleft gf\alpha_k\mu)(gf\mu l) \\
&= (gd\sigma)(gf\sigma)[(g\theta_d k \triangleleft gf\mu\rho)gf\mu k][(g\theta_d l \triangleleft gf\mu\theta_k)gf\mu l] \\
&= (gd\sigma)(gf\sigma)(g\theta_d gf\mu)k(g\theta_d gf\mu)l \\
&= (gd\sigma)(gf\sigma)(g\theta_f k)(g\theta_f l) \\
&= [g(d\cdot f)\sigma][g\theta_f(k\cdot l)] \\
&= g[(d\cdot f)*(k\cdot l)]
\end{aligned}$$

as required.

Chapter 7

Ordered crossed modules

7.1 Derivations on ordered groupoids

Although we shall mostly be concerned with crossed modules of groupoids in this chapter, we can take the first few steps for any ordered groupoid action on an ordered groupoid. We follow Brown's formulation [4] of the cohomology $H^1(G, A)$. We note that, in this chapter, d and r will be the domain and range maps for groupoids, and that derivations will be denoted by Greek letters $\delta, \eta, \ldots$ etc.

So let G be an ordered groupoid acting on an ordered groupoid A and let $\theta : G \to G$ be an ordered functor. We define the ordered groupoid $Z^1_\theta(G, A)$ to be the fibre over θ of the morphism

$$p_* : \mathsf{OGPD}(G, G \ltimes A) \to \mathsf{OGPD}(G, G)$$

induced by the projection $G \ltimes A \to G$. Recall from Propositions 4.2.1 and 3.3.1 that p_* is a strong fibration.

An object of $Z^1_\theta(G, A)$ is an ordered functor $G \to G \ltimes A$ of the form $g \mapsto (g\theta, g\delta)$ where $\delta : G \to A$ is an ordered function that satisfies

- $g\delta w = r(g\theta)$,
- if $g \in G(x, y)$ then $g\delta \in A(x\delta \lhd g\theta, y\delta)$,
- if g, h are composable in G then $(gh)\delta = (g\delta \lhd h\theta)(h\delta)$.

These properties define δ as a *θ–derivation* $G \to A$, generalising the case for a G–module M considered in chapter 5. We shall therefore identify the set of objects of $Z^1_\theta(G, A)$ with the set $\mathrm{Der}_\theta(G, A)$ of θ–derivations $G \to A$.

Proposition 7.1.1. *Let δ, η be θ–derivations $G \to A$. Then the set of arrows in $Z^1_\theta(G, A)$*

from δ to η is bijective with the set of ordered mappings $\phi : \mathrm{Ob}(G) \to A$ such that for all $x, y \in \mathrm{Ob}(G)$,

1. *$x\phi w = x\theta$,*
2. *$x\phi \in A(x\delta, x\eta)$,*
3. *for $g \in G(x, y)$ we have $(x\phi \lhd g\theta)g\eta = (g\delta)(y\phi)$.*

Proof. An arrow in $Z^1_\theta(G, A)$ from δ to η is an ordered natural transformation between the corresponding functors that projects to the identity natural transformation at θ under p_*. It is therefore given by an ordered function $\mathrm{Ob}(G) \to G \ltimes A$ of the form $x \mapsto (x\theta, x\phi)$ for some ordered function $\phi : \mathrm{Ob}(G) \to A$, such that for all $g \in G(x, y)$, the following square commutes:

$$\begin{tikzcd}
(x\theta, x\delta) \arrow[r, "(g\theta, g\delta)"] \arrow[d, "(x\theta, x\phi)"'] & (y\theta, y\delta) \arrow[d, "(y\theta, y\phi)"] \\
(x\theta, x\eta) \arrow[r, "(g\theta, g\eta)"'] & (y\theta, y\eta)
\end{tikzcd}$$

Therefore, in $G \ltimes A$ we have

$$(g\theta, (x\phi \lhd g\theta)g\eta) = (x\theta, x\phi)(g\theta, g\eta) = (g\theta, g\delta)(y\theta, y\phi) = (g\theta, g\delta \cdot y\phi)\,.$$

□

Proposition 7.1.2. *For any θ–derivation $\delta : G \to A$, each ordered function $\phi : \mathrm{Ob}(G) \to A$ such that $\phi w = \theta|_{\mathrm{Ob}(G)}$ and $d(x\phi) = x\delta$ determines an arrow in $Z^1_\theta(G, A)$ starting at δ.*

Proof. For $g \in G(x, y)$ we define $\eta : G \to A$ by $g\eta = (x\phi \lhd g\theta)^{-1}(g\delta)(y\phi)$. Then $g\eta w = y\phi w = y\theta = r(g)\theta = r(g\theta)$, $g\eta \in A(r(x\phi \lhd g\theta), r(y\phi)) = A(x\eta \lhd g\theta, y\eta)$, and for $h \in G(y, z)$ we have

$$\begin{aligned}
(gh)\eta &= (x\phi \lhd (gh)\theta)^{-1}((gh)\delta)(z\phi) \\
&= (x\phi \lhd (gh)\theta)^{-1}(g\delta \lhd h\theta)(h\delta)(z\phi) \\
&= (x\phi \lhd (gh)\theta)^{-1}(g\delta \lhd h\theta)(y\phi \lhd h\theta)(y\phi \lhd h\theta)^{-1}(h\delta)(z\phi) \\
&= [(x\phi \lhd g\theta)^{-1}(g\delta)(y\phi)] \lhd h\theta \cdot (y\phi \lhd h\theta)^{-1}(h\delta)(z\phi) \\
&= (g\eta \lhd h\theta)(h\eta)
\end{aligned}$$

and η is a θ–derivation. □

The groupoid structure on $Z^1_\theta(G, A)$ may be described explicitly using the identifications we have made. The set of objects of $Z^1_\theta(G, A)$ is the set of $\mathrm{Der}_\theta(G, A)$ of ordered θ–derivations $G \to A$. An arrow from δ to η is a pair (δ, ϕ) where ϕ is an ordered mapping $\mathrm{Ob}(G) \to A$ such that for all $x \in \mathrm{Ob}(G)$ we have $x\phi w = x\theta$ and $d(x\phi) = x\delta$. (Hence ϕ determines δ on $\mathrm{Ob}(G)$.) The derivation η is then given by $\eta : g \mapsto (d(g)\phi \lhd g\theta)^{-1}(g\delta)(r(g)\phi)$, and the composition of arrows is given by

$$(\delta, \phi)(\eta, \psi) = (\delta, \phi + \psi)$$

where $x(\phi + \psi) = x\phi \cdot x\psi$.

Continuing with these identifications, the kernel of the strong fibration p_* is the union $\bigsqcup_{\theta:G\to G} Z^1_\theta(G, A)$.

Example 7.1.3. Let P is a poset acted on by G, and take $\theta = \mathrm{id}_G$. Then an ordered derivation $\delta : G \to P$ is an ordered function that satisfies $\delta w = \mathrm{id}_{\mathrm{Ob}(G)}$, and for all $g \in G(x, y)$, $g\delta = y\delta = x\delta \lhd g$. $Z^1(G, P)$ is also a poset, with the functions $G \to P$ ordered pointwise.

7.2 Ordered crossed modules

We shall now look in more detail at the case of an ordered crossed module, and as in chapter 6, we shall find more structure. The ideas in the unordered case originate in [8] and [6], and were set out in detail for crossed modules of groupoids in [11]. Recall from chapter 5 that an *ordered crossed module* consists of an ordered functor $\mu : C \to G$ of ordered groupoids that is the identity on the sets of objects, with C a disjoint union of its vertex groups, and an ordered action of G on C with $w : C \to Ob(G)$ equal to the identity on objects, and in addition satisfying

CM1 $(c \lhd g)\mu = g^{-1}(c\mu)g$,

CM2 $c \lhd (a\mu) = a^{-1}ca$,

for all $a, c \in C(x)$ and $g \in G(x, y)$.

We shall usually denote a crossed module as above by (C, G, μ). If only [CM1] is satisfied we have a *precrossed module*.

A morphism of ordered crossed modules $\theta : (C, G, \mu) \to (C', G', \mu')$ is a pair $\theta = \langle\theta_1, \theta_2\rangle$, where θ_1 is an ordered morphism of groupoids $G \to G'$ and θ_2 is an ordered morphism of groupoids $C \to C'$, such that (recalling that $\mathrm{Ob}(G) = \mathrm{Ob}(C)$),

- $\theta_1|_{\mathrm{Ob}(G)} = \theta_2|_{\mathrm{Ob}(C)}$,
- $\theta_2\mu' = \mu\theta_1$,

- $(c \lhd g)\theta_2 = c\theta_2 \lhd g\theta_1$.

We shall often omit the subscript i on θ_i where no confusion can arise.

We denote the category of ordered crossed modules by $\mathcal{OCM}$.

If (C, G, μ) is an ordered crossed G–module and θ is an ordered morphism $(C, G, \mu) \to (C, G, \mu)$ then an *ordered homotopy* (compare Brown and İçen [11]) on θ is a pair of ordered functions $\delta = (\delta_0, \delta_1)$ such that $\delta_0 : \mathrm{Ob}(G) \to G$ satisfies $d(x\delta_0) = x\theta$ and $\delta_1 : G \to C$ is a θ–derivation. Hence δ_1 satisfies:

- if $g \in G(x, y)$ then $g\delta_1 \in C(y\theta)$,
- if g, h are composable in G then $(gh)\delta_1 = (g\delta_1 \lhd h\theta)(h\delta_1)$.

This is part of the more general theory of homotopy within the monoidal closed structure for crossed complexes set out in [8]. At the conclusion of this chapter we shall look at the cartesian closed structure, following Howie [19]. Our first result is the ordered version of [11, Proposition 2.2]: the necessary verifications follow those in [11] but we give the details for the reader's convenience. This result generalises Lemma 6.3.3.

Proposition 7.2.1. *If (C, G, μ) is an ordered crossed module and δ is an ordered homotopy on θ then, for $g \in G(x, y)$ and $c \in C(x)$, the functions*

$$\xi_1 : g \mapsto (x\delta_0)^{-1} g\theta(g\delta_1\mu)(y\delta_0)\,,$$
$$\xi_2 : c \mapsto ((c\theta)(c\mu\delta_1)) \lhd x\delta_0$$

determine an ordered morphism $\xi = \langle \xi_1, \xi_2 \rangle$ of crossed modules.

Proof. It is clear that ξ will be ordered. For $x \in \mathrm{Ob}(G) = \mathrm{Ob}(C)$ we have

$$x\xi_1 = (x\delta_0)^{-1} x\theta x\delta_1\mu x\delta_0 = [(x\theta)(x\delta_1) \lhd x\delta_0]\mu = (x\theta)(x\delta_1) \lhd x\delta_0 = x\xi_2$$

since μ is the identity on $\mathrm{Ob}(G) = \mathrm{Ob}(C)$, and so $x\theta = x\theta\mu = x\mu\theta$.

For composable arrows $g \in G(x, y)$ and $h \in G(y, z)$ we have

$$\begin{aligned}
g\xi_1 h\xi_1 &= (x\delta_0)^{-1} g\theta(g\delta_1\mu)(y\delta_0)(y\delta_0)^{-1} h\theta(h\delta_1\mu)(z\delta_0) \\
&= (x\delta_0)^{-1} g\theta h\theta(h\theta)^{-1}(g\delta_1\mu)h\theta(h\delta_1\mu)(z\delta_0) \\
&= (x\delta_0)^{-1}(gh)\theta(g\delta_1 \lhd h\theta)\mu(h\delta_1\mu)(z\delta_0) \\
&= (x\delta_0)^{-1}(gh)\theta((g\delta_1 \lhd h\theta)(h\delta_1))\mu(z\delta_0) \\
&= (x\delta_0)^{-1}(gh)\theta(gh)\delta_1\mu(z\delta_0) \\
&= (gh)\xi_1
\end{aligned}$$

and for $ab \in C(x)$,

$$\begin{aligned}
a\xi_2 b\xi_2 &= ((a\theta)(a\mu\delta_1)) \lhd x\delta_0((b\theta)(b\mu\delta_1)) \lhd x\delta_0 \\
&= ((a\theta)(a\mu\delta_1)(b\theta)(b\mu\delta_1)) \lhd x\delta_0 \\
&= ((a\theta)(b\theta)(b\theta)^{-1}(a\mu\delta_1)(b\theta)(b\mu\delta_1)) \lhd x\delta_0 \\
&= ((ab)\theta(a\mu\delta_1) \lhd (b\theta\mu)(b\mu\delta_1)) \lhd x\delta_0 \\
&= ((ab)\theta(a\mu\delta_1) \lhd (b\mu\theta)(b\mu\delta_1)) \lhd x\delta_0 \\
&= ((ab)\theta(ab)\mu\delta_1)) \lhd x\delta_0 \\
&= (ab)\xi_2.
\end{aligned}$$

Hence ξ_1 and ξ_2 are ordered functors that agree on the object set $\mathrm{Ob}(G)$. Now for $c \in C(x)$ we have

$$\begin{aligned}
c\mu\xi_1 &= (x\delta_0)^{-1}c\mu\theta(c\mu\delta_1\mu)(x\delta_0) \\
&= (x\delta_0)^{-1}c\theta\mu(c\mu\delta_1\mu)(x\delta_0) \\
&= (x\delta_0)^{-1}(c\theta(c\mu\delta_1))\mu(x\delta_0) \\
&= [(c\theta(c\mu\delta_1)) \lhd x\delta_0]\mu \\
&= c\xi_2\mu\,.
\end{aligned}$$

Finally, for $c \in C(x)$ and $g \in G(x,y)$ we have

$$\begin{aligned}
(c \lhd g)\xi_2 &= [(c \lhd g)\theta(c \lhd g)\mu\delta_1] \lhd y\delta_0 \\
&= [(c\theta \lhd g\theta)(g^{-1}(c\mu)g)\delta_1] \lhd y\delta_0 \\
&= [(c\theta \lhd g\theta)(g^{-1}(c\mu))\delta_1 \lhd g\theta)g\delta_1] \lhd y\delta_0 \\
&= [(g\delta_1) \lhd (g^{-1}\theta) \cdot (c\theta)(g^{-1}(c\mu))\delta_1)] \lhd (g\theta)(g\delta_1\mu)(y\delta_0) \\
&= [(g\delta_1) \lhd (g^{-1}\theta) \cdot (c\theta)[(g^{-1}\delta_1) \lhd (c\mu\theta)](c\mu\delta_1)] \lhd (g\theta)(g\delta_1\mu)(y\delta_0) \\
&= [((g\delta_1) \lhd (g^{-1}\theta) \cdot (c\theta)[(g^{-1}\delta_1) \lhd c\theta\mu](c\mu\delta_1)] \lhd (g\theta)(g\delta_1\mu)(y\delta_0) \\
&= [(g\delta_1) \lhd (g^{-1}\theta) \cdot (c\theta)(c\theta)^{-1}(g^{-1}\delta_1)(c\theta)(c\mu\delta_1)] \lhd (g\theta)(g\delta_1\mu)(y\delta_0) \\
&= [(c\theta)(c\mu\delta_1)] \lhd (x\delta_0) \lhd (x\delta_0)^{-1}(g\theta)(g\delta_1\mu)(y\delta_0) \\
&= c\xi_2 \lhd g\xi_1\,.
\end{aligned}$$

This concludes the verifications that ξ is a crossed module morphism. $\square$

The monoidal closed structure on the category of crossed complexes [8] implies that, for any crossed complex C, there exists a crossed complex $\mathrm{END(C)}$ that is an internal monoid in the category of crossed complexes. If (C, G, μ) is an crossed module, then so is $\mathsf{END}(C, G, \mu)$ and the content of Proposition 7.2.1 is just the verification of properties of the target map $r : \mathsf{END}_1 \to \mathsf{END}_0$. Because $\mathsf{END}(C, G, \mu)$ is an internal monoid, it carries some extra structure, which was described in [6] using the idea of a *braided, semiregular* crossed module, We shall now describe some further details of this structure, which we can use to prove generalisations of some results in [11]. All of the details are extracted from [8].

7.2.1 The ordered crossed module $\mathsf{END}(C, G, \mu)$

We write $\mathsf{END}(C, G, \mu)$ as $(\mathcal{E}_2, \mathcal{E}_1, \nu)$: its object set is the monoid $\mathrm{End}(C, G, \mu)$ of ordered endomorphisms of (C, G, μ). Arrows in $\mathcal{E}_1$ are ordered homotopies: an ordered homotopy δ on $\theta \in \mathrm{End}(C, G, \mu)$ has $d(-\delta) = \theta$ and $r(-\delta) = \xi$ where ξ is defined in Proposition 7.2.1. We write $\delta : \theta \Longrightarrow \xi$. If $\lambda : \xi \Longrightarrow \psi$ then the composition $\delta\lambda \in \mathcal{E}_1$ is defined as follows.

For $x \in \mathrm{Ob}(G)$ we have

$$x(\delta\lambda)_0 = (x\delta_0)(x\lambda_0) \in G\,.$$

$x\delta_0$ and $x\lambda_0$ are composable in G since $d(x\lambda_0) = x\xi = r(x\delta_0)$.

For $g \in G(x, y)$ we have

$$g(\delta\lambda)_1 = g\delta_1(g\lambda_1 \lhd y\delta_0^{-1}) \in C\,. \tag{7.2.1}$$

This composition is defined since $g\lambda_1 \in C(y\xi_1) = C(r(y\delta_0))$ and so $g\lambda_1 \lhd y\delta_0^{-1} \in C(d(y\delta_0)) = C(y\theta)$ and $g\delta_1 \in C(y\theta)$ too.

Then $\delta\lambda : \theta \Longrightarrow \psi$ certainly $d(x(\delta\lambda)_0) = d(x\delta_0) = x\theta$, and $g(\delta\lambda)_1 \in C(y\theta)$. Then if $h \in G(y, z)$,

$$\begin{aligned}
(gh)(\delta\lambda)_1 &= (gh)\delta_1((gh)\lambda_1 \lhd z\delta_0^{-1}) \\
&= (g\delta_1 \lhd h\theta)(h\delta_1)[(g\lambda_1 \lhd h\xi)(h\lambda_1)] \lhd z\delta_0^{-1} \\
&= (g\delta_1 \lhd h\theta)(h\delta_1)(g\lambda_1 \lhd (y\delta_0^{-1}h\theta h\delta_1\mu))((h\lambda_1) \lhd z\delta_0^{-1}) \\
&= (g\delta_1 \lhd h\theta)(g\lambda_1 \lhd (y\delta_0^{-1}h\theta))(h\delta_1)((h\lambda_1) \lhd z\delta_0^{-1}) \\
&= ((g\delta_1)(g\lambda_1 \lhd (y\delta_0^{-1})) \lhd h\theta)(h\delta_1)((h\lambda_1) \lhd z\delta_0^{-1}) \\
&= (g(\delta\lambda)_1 \lhd h\theta)h(\delta\lambda)_1
\end{aligned}$$

and so $\delta\lambda$ is a homotopy on θ. A similar calculation verifies that $r(-\delta\lambda) = \psi$, and the inverse of δ is $\overline{\delta}$, where $x\overline{\delta}_0 = (x\delta_0)^{-1}$ and

$$g\overline{\delta}_1 = (g\delta_1 \lhd y\overline{\delta_0})^{-1}\,.$$

This structure for $\mathcal{E}_1$ generalises the one for $Der_\diamond(G, A)$ in Theorem 6.3.13.

Arrows in $\mathcal{E}_2$ are, in the terminology of [8], 2–fold homotopies $C \to C$. For a crossed module (C, G, μ), the elements of $\mathcal{E}_2(\theta)$ are ordered functions $\sigma : \mathrm{Ob}(G) \to C$, such that for all $x \in \mathrm{Ob}(G)$ we have $x\sigma \in C(x\theta)$. $\mathcal{E}_2(\theta)$ is a group under pointwise multiplication. The ordered functor $\nu : \mathcal{E}_2 \to \mathcal{E}_1$ maps $\sigma \in \mathcal{E}_2(\theta)$ to the ordered homotopy $\sigma\nu = (\sigma\mu, \delta_\sigma)$ where, for $g \in G(x, y)$,

$$g\delta_\sigma = ((x\sigma) \lhd g\theta)(y\sigma)^{-1} .$$

$\sigma\nu$ is an ordered homotopy on θ, since for any $x \in \mathrm{Ob}(G)$ and $g \in G(x, y), h \in G(y, z)$,

- $d(x\sigma\mu) = x\theta\mu = x\theta,$
- $g\delta_\sigma \in C(y\sigma) = C(y\theta),$
- $$\begin{aligned}(gh)\delta_\sigma &= (x\sigma \lhd (gh)\theta)(z\sigma)^{-1}\\ &= [(x\sigma \lhd g\theta)(y\sigma)^{-1}] \lhd h\theta \cdot (y\sigma \lhd h\theta)(z\sigma)^{-1}\\ &= (g\delta_\sigma \lhd h\theta)h\delta_\sigma.\end{aligned}$$

Then if $\sigma\nu : \theta \Longrightarrow \xi$ with ξ given in Proposition 7.2.1, we have for $g \in G(x, y)$,

$$\begin{aligned}g\xi &= (x\sigma\mu)^{-1} \cdot g\theta \cdot g\delta_\sigma\mu \cdot y\sigma\mu\\ &= (x\sigma\mu)^{-1} \cdot g\theta \cdot (x\sigma \lhd g\theta \cdot y\sigma^{-1})\mu \cdot y\sigma\mu\\ &= (x\sigma\mu)^{-1} \cdot g\theta(x\sigma \lhd g\theta)\mu = g\theta\,,\end{aligned}$$

and for $c \in C(x)$,

$$\begin{aligned}c\xi &= (c\theta \cdot c\mu\delta_\sigma) \lhd x\sigma\mu\\ &= [c\theta \cdot x\sigma \lhd c\mu\theta \cdot x\sigma^{-1}] \lhd x\sigma\mu\\ &= [x\sigma \cdot c\theta \cdot x\sigma^{-1}] \lhd x\sigma\mu = c\theta\,,\end{aligned}$$

and so $\xi = \theta$. Hence ν gives a family of maps $\mathcal{E}_2(\theta) \to \mathcal{E}_1(\theta)$.

Now $\nu : \mathcal{E}_2 \to \mathcal{E}_1$ is a crossed module. Firstly, if $\sigma, \tau \in \mathcal{E}_2(\theta)$ then $(\sigma \cdot \tau)\nu$ is the ordered homotopy $((\sigma \cdot \tau)\mu, \delta_{\sigma\cdot\tau})$, where $(\sigma \cdot \tau)\mu = \sigma\mu \cdot \tau\mu$ and

$$\begin{aligned}g\delta_{\sigma\cdot\tau} &= ((x\sigma)(x\tau)) \lhd g\theta)((y\sigma)(y\tau))^{-1}\\ &= (x\sigma \lhd g\theta)(y\sigma)^{-1}(y\sigma)(x\tau \lhd g\theta)(y\tau)^{-1}(y\sigma)^{-1}\\ &= (x\sigma \lhd g\theta)(y\sigma)^{-1}[(x\tau \lhd g\theta)(y\tau)^{-1}] \lhd (y\sigma)^{-1}\mu\\ &= g\delta_\sigma \cdot (g\delta_\tau) \lhd (y\sigma\mu)^{-1}\\ &= g(\delta_\sigma * \delta_\tau),\end{aligned}$$

and so $\nu : \mathcal{E}_2(\theta) \to \mathcal{E}_1(\theta)$ is a group homomorphism.

The action of $\delta \in \mathcal{E}_1(\theta, \xi)$ on $\sigma \in \mathcal{E}_2(\theta)$ is given by $x(\sigma \triangleleft \delta) = x\sigma \triangleleft x\delta_0$. Then $x\sigma \triangleleft x\delta_0 \in C(r(x\delta_0)) = C(x\xi)$ as required.

We check the crossed module rules CM1 and CM2. So for $\sigma, \tau \in \mathcal{E}_2(\theta)$ and $\eta \in \mathcal{E}_1(\theta, \xi)$ we have, for $x \in \mathrm{Ob}(G)$,

$$x(\eta(\sigma \triangleleft \eta)\nu)_0 = (x\eta_0)(x\sigma \triangleleft x\eta)\mu = x\sigma\mu \cdot x\eta_0 = x(\sigma\nu)\eta$$

and for $g \in G(x, y)$,

$$\begin{aligned}
g(\eta\delta_{\sigma\triangleleft\eta})_1 &= g\eta \cdot g\delta_{\sigma\triangleleft\eta} \triangleleft y\eta^{-1} \\
&= g\eta \cdot [(x\sigma \triangleleft x\eta) \triangleleft g\xi \cdot (y\sigma \triangleleft y\eta)^{-1}] \triangleleft y\eta^{-1} \\
&= g\eta \cdot [(x\sigma \triangleleft x\eta) \triangleleft (x\eta^{-1}g\theta \cdot g\eta\mu \cdot y\eta) \cdot (y\sigma)^{-1} \triangleleft y\eta)] \triangleleft y\eta^{-1} \\
&= g\eta \cdot (x\sigma \triangleleft g\theta \triangleleft g\eta\mu) \cdot (y\sigma)^{-1} \\
&= (x\sigma \triangleleft g\theta) \cdot g\eta \cdot (y\sigma)^{-1}
\end{aligned}$$

whereas

$$\begin{aligned}
g(\sigma\nu\eta)_1 &= g(\delta_\sigma\eta) \\
&= g\delta_\sigma \cdot g\eta \triangleleft (y\sigma\mu)^{-1} \\
&= (x\sigma \triangleleft g\theta)(y\sigma)^{-1} \cdot g\eta \triangleleft (y\sigma\mu)^{-1} \\
&= (x\sigma \triangleleft g\theta) \cdot g\eta \cdot (y\sigma)^{-1}
\end{aligned}$$

These calculations show that $\eta(\sigma \triangleleft \eta)\nu = (\sigma\nu)\eta$ and so CM1 holds. For CM2,

$$x(\sigma \triangleleft \tau\nu) = x\sigma \triangleleft x\tau\mu = (x\tau)^{-1} \cdot x\sigma \cdot x\tau = x(\overline{\tau}\sigma\tau)\,.$$

In the groups case considered in chapter 6, from a crossed module (A, G, μ) we still get a crossed module of groupoids

$$\mathcal{E}_2 \xrightarrow{\nu} \mathcal{E}_1 \rightrightarrows \mathrm{End}(A, G, \mu)$$

where $\mathcal{E}_1 = Der_\diamond(G, A)$ (see Theorem 6.3.13) and

$$\mathcal{E}_2 = \bigsqcup_{\theta \in \mathrm{End}(A,G,\mu)} A_\theta$$

is a disjoint union of copies of A indexed by $\mathrm{End}(A, G, \mu)$. If $a \in A_\theta$ then

$$a\nu : g \mapsto (a \lhd g\theta)a^{-1}$$

is a derivation $G \to A$ linked to θ.

7.2.2 Semiregular crossed modules

According to [6, Theorem 1.3] a crossed complex is an internal monoid in the monoidal category of crossed complexes if and only if it is *braided* and *semiregular*. A crossed module (C, G, μ) with object set $B = \mathrm{Ob}(C) = \mathrm{Ob}(G)$ is semiregular if B is a monoid and there are commuting left and right monoid actions of B on C and and on G (denoted by $(b, c) \mapsto b \wr c$ and so on), such that

- each action of B preserves the groupoid structure of G, and the source and target maps $G \to B$ are B–equivariant (where B acts on itself by left and right multiplication),
- each action of B preserves the group structures in C, and if $c \in C_2(x)$ and $y \in B$ then $y \wr c \in C_2(yx)$ and $c \wr y \in C_2(xy)$,
- if $c \in C_2(x)$ and $g \in G(x, y)$ then for all $b \in B$ we have

  $$b \wr (c \lhd g) = (b \wr c) \lhd (b \wr g) \in C_2(by) \text{ and } (c \lhd g) \wr b = (c \wr b) \lhd (g \wr b) \in C_2(yb)\,,$$

- μ is both left and right B–equivariant.

In particular, both C and G are semiregular categories in the sense of [15], and from that paper we have:

Proposition 7.2.2. *([15, Proposition 1.1]) If G is a semiregular category with object set B, then G is a monoid with binary operation*

$$g * h = (g \wr d(h))(r(g) \wr h)$$

and identity element the identity arrow 1_e at the identity e of the monoid B.

Corollary 7.2.3. $\mathrm{star}_G(e) = \{g \in G : d(g) = e\}$ *is a submonoid of* $(G, *)$.

Corollary 7.2.4. *If (C, G, μ) is a semiregular crossed module with $\mathrm{Ob}(C) = \mathrm{Ob}(G) = B$, and if e is the identity of the monoid B, then the maps $\mu : C_2(e) \to \mathrm{star}_G(e)$ and $r : \mathrm{star}_G(e) \to B$ are monoid homomorphisms.*

Semiregular categories and their applications are discussed in [5] under the name of *whiskered* categories. In a semiregular category G, if $\mathrm{Ob}(G) = B$ is a group then G is said to be *regular*. If G is a regular groupoid then the binary operation $*$ of Proposition 7.2.2 makes G into a group (see [15, Proposition 1.3(i)]), with $g \in (G, *)$ having inverse $\overline{g}^* = r(g)^{-1} \wr g^{-1} \wr d(g)^{-1}$, and with a B–action by automorphisms given by $g \lhd b = b^{-1} \wr g \wr b$. Then $\mathrm{star}_G(e)$ is a B–invariant subgroup and $r : \mathrm{star}_G(e) \to B$ is a *precrossed module* – that is, a homomorphism (of groups) satisfying CM1.

If (C, G, μ) is a regular crossed module then both C and G are regular categories. Then $C(e)$ is a subgroup of $(C, *)$ and since e acts trivially in the $\wr$–actions, the group operation $*$ on $C(e)$ is equal to the given group operation in (C, G, μ). Then $\mu : C(e) \to \mathrm{star}_G(e)$ is a crossed module of groups, with a new composite action of $\mathrm{star}_G(e)$ on $C(e)$ given by

$$c^g = r(g)^{-1} \wr (c \lhd g)\,.$$

The properties CM1 and CM2 are easy to verify for this new action. For CM1 we have

$$\begin{aligned}(c^g)\mu &= [r(g)^{-1} \wr c \lhd g]\mu = r(g)^{-1} \wr (c \lhd g)\mu \\ &= r(g)^{-1} \wr (g^{-1}(c\mu)g) \\ &= r(g)^{-1} \wr g^{-1} \cdot r(g)^{-1} \wr c\mu \cdot r(g)^{-1} \wr g \\ &= \overline{g}^* * c\mu * g\end{aligned}$$

and for CM2, for $a, c \in C(e)$ we have

$$c^{a\mu} = r(a\mu)^{-1} \wr c \lhd a\mu = e \wr a^{-1}ca = a^{-1}ca = \overline{a}^* * c * a\,.$$

7.2.3 Application to $\mathsf{END}(C, G, \mu)$

In the semiregular crossed module $\mathsf{END}(C, G, \mu)$, the monoid $\mathrm{End}(C, G, \mu)$ acts on $\mathcal{E}_1$ and $\mathcal{E}_2$ by left and right composition of mappings. It is therefore possible to give detailed formulae for the compositions in $\mathcal{E}_1$ and $\mathcal{E}_2$ implied by Proposition 7.2.2.

If $\delta : \theta \Longrightarrow \xi$ and $\eta : \phi \Longrightarrow \psi$ then $\delta * \eta : \theta\phi \Longrightarrow \xi\psi$ is defined for $x \in \mathrm{Ob}(G)$ and $g \in G(x, y)$ by

$$(\delta * \eta)_0 = (\delta_0 \wr \phi)(\xi \wr \eta_0) : x \mapsto (x\delta_0\phi)(x\xi\eta_0) \tag{7.2.2}$$

and

$$(\delta * \eta)_1 = (\delta_1 \wr \phi)(\xi \wr \eta_1) : g \mapsto (g\delta_1\phi)((g\xi\eta_1 \lhd (y\delta_0\phi)^{-1}) \tag{7.2.3}$$

by equation (7.2.1). These can be expanded further using the formulae for ξ given in Proposition 7.2.1.

If $\sigma \in \mathcal{E}_2(\theta)$ and $\tau \in \mathcal{E}_2(\phi)$ then for $x \in \mathrm{Ob}(G)$ we have

$$(\sigma * \tau) = (\sigma \wr \phi) \cdot (\theta \wr \tau) : x \mapsto (x\sigma\phi)(x\theta\tau) \in C(x\theta\phi)\,.$$

If we restrict to $\mathrm{star}_{\mathcal{E}_1}(\mathrm{id})$, which is a submonoid of $(\mathcal{E}_1, *)$ by Corollary 7.2.3, then for $\delta, \eta \in \mathrm{star}_{\mathcal{E}_1}(\mathrm{id})$ we get

$$(\delta * \eta)_1 = (\delta_1 \wr \phi)(\xi \wr \eta_1) : g \mapsto (g\delta_1)((g\xi\eta_1 \lhd (y\delta_0)^{-1}) \tag{7.2.4}$$

and this (allowing for changes of convention) is the monoid operation given in [11, Proposition 2.4]. In fact, Brown and İçen [11] consider the subcrossed module $\mathcal{A}$ of $\mathcal{E} = \mathsf{END}(C, G, \mu)$ in whch $\mathcal{A}_0 = \mathrm{Aut}(C, G, \mu)$, $\mathcal{A}_1$ is the full groupoid of $\mathcal{E}_1$ spanned by $\mathcal{A}_0$, and $\mathcal{A}_2 \subseteq \mathcal{E}_2$. Then $\mathcal{A}$ is a regular groupoid, and so $r : \mathrm{star}_{\mathcal{A}}(\mathrm{id}) \to \mathrm{Aut}(C, G, \mu)$ is a precrossed module ([11, Theorem 2.6]) and $\mathcal{A}_2(\mathrm{id}) \to \mathrm{star}_{\mathcal{A}}(\mathrm{id})$ is a crossed module ([11, Theorem 2.8]).

7.3 The cartesian closed case

As part of a wide-ranging study of pullbacks for crossed complexes, Howie [19] describes a cartesian closed structure on the category of crossed modules, that also restricts to the ordered case, giving an internal hom-functor OCM on $\mathcal{OCM}$. We briefly describe Howie's construction.

Let (C, G, μ) and (C', G', μ') be ordered crossed modules. The object set of the ordered crossed module

$$\mathrm{OCM}((C, G, \mu), (C', G', \mu'))$$

is the set of ordered crossed module morphisms $(C, G, \mu) \to (C', G', \mu')$, and given two such morphisms $\theta = \langle \theta_1, \theta_2 \rangle$ and $\xi = \langle \xi_1, \xi_2 \rangle$ an arrow from θ to ξ is an ordered natural transformation $\eta : \theta \Longrightarrow \xi$ that also satisfies $c\xi_2 = (c\theta_2) \lhd x\eta$, for all $c \in C_2(x)$. The group $\mathrm{OCM}_2(\theta)$ consists of all ordered functions $\zeta : \mathrm{Ob}(C) \to C'_2$ such that $x\zeta \in C'_2(x\theta)$ and for all $g \in G(x, y)$ we have $y\zeta = (x\zeta) \lhd g\theta$. We define $\nu : \mathrm{OCM}_2(\theta) \to \mathrm{OCM}_1(\theta)$ by $\zeta\nu = (\theta, \zeta\mu')$. Then $\zeta\mu'$ is a natural transformation from θ to θ since for $g \in G(x, y)$,

$$\begin{aligned}(x\zeta\mu')^{-1}(g\theta)(y\zeta\mu') &= (x\zeta\mu')^{-1}(g\theta)((x\zeta \lhd g\theta)\mu') \\ &= (x\zeta\mu')^{-1}(g\theta)(g\theta)^{-1}(x\zeta\mu')(g\theta) \\ &= g\theta\,.\end{aligned}$$

Proposition 7.3.1. $\mathrm{OCM}((C,G,\mu),(C',G',\mu'))$ *is an ordered crossed module.*

Proof. Most of the facts can be deduced from [19], but we give some further details here. The groupoid $\mathrm{OCM}_1((C,G,\mu),(C',G',\mu'))$ is an ordered subgroupoid of $\mathsf{OGPD}(C \times G, C' \times G')$, and $\mathrm{OCM}_2(\theta)$ is a group under pointwise composition (denoted by $\zeta_1 \cdot \zeta_2$). If $\theta \leqslant \theta'$ then for $\zeta \in \mathrm{OCM}_2(\theta')$ we define the restriction $(\theta|\zeta)$ by $x(\theta|\zeta) = (x\theta|x\zeta)$. The action of a natural transformation η on $\zeta \in \mathrm{OCM}_2(\theta)$ is defined via the action of G' on C' by $x(\zeta \lhd (\theta,\eta)) = x\zeta \lhd x\eta$.

We now verify the properties CM1 and CM2. For CM1, take $\zeta \in \mathrm{OCM}_2(\theta)$. Then $(\zeta \lhd (\theta,\eta))\nu$ is a natural transformation whose component at $x \in \mathrm{Ob}(G)$ is

$$(x\zeta \lhd x\eta)\mu' = (x\eta)^{-1}x\zeta\mu'(x\eta)\,,$$

and this is exactly the x–component of the natural transformation in the composition $(\theta,\eta)^{-1}\zeta\nu(\theta,\eta)$. Hence CM1 holds. For CM2, take $\zeta, \rho \in \mathrm{OCM}_2(\theta)$. Then

$$x(\zeta \lhd \rho\nu) = x\zeta \lhd x\rho\nu = (x\rho)^{-1}(x\zeta)(x\rho) = x(\rho^{-1} \cdot \zeta \cdot \rho)$$

and CM2 holds. □

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